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# HYPERFUNCTION SOLUTIONS TO INVARIANT DIFFERENTIAL EQUATIONS ON THE SPACE OF REAL SYMMETRIC MATRICES

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**ABSTRACT.** The real special linear group of degree  $n$  naturally acts on the vector space of  $n \times n$  real symmetric matrices. How to determine invariant hyperfunction solutions of invariant linear differential equations with polynomial coefficients on the vector space of  $n \times n$  real symmetric matrices is discussed in this paper. We observe that every invariant hyperfunction solution is expressed as a linear combination of Laurent expansion coefficients of the complex power of the determinant function with respect to the parameter of the power. Then the problem is reduced to the determination of Laurent expansion coefficients which is needed to express. We give an algorithm to determine them and apply the algorithm in some examples.

## INTRODUCTION.

Let  $V := \text{Sym}_n(\mathbb{R})$  be the space of  $n \times n$  symmetric matrices over the real field  $\mathbb{R}$  and let  $\text{SL}_n(\mathbb{R})$  be the special linear group over  $\mathbb{R}$  of degree  $n$ . Then the group  $G := \text{SL}_n(\mathbb{R})$  acts on the vector space  $V$  by the representation

$$\rho(g) : x \longmapsto g \cdot x := gx^t g, \quad (1)$$

with  $x \in V$  and  $g \in G$ . Let  $D(V)$  be the algebra of linear differential operators on  $V$  with polynomial coefficients and let  $\mathcal{B}(V)$  be the space of hyperfunctions on  $V$ . We denote by  $D(V)^G$  and  $\mathcal{B}(V)^G$  the subspaces of  $G$ -invariant linear differential operators and of  $G$ -invariant hyperfunctions on  $V$ , respectively. For a given invariant differential operator  $P(x, \partial) \in D(V)^G$  and an invariant hyperfunction  $v(x) \in \mathcal{B}(V)^G$ , we consider the linear differential equation

$$P(x, \partial)u(x) = v(x) \quad (2)$$

where the unknown function  $u(x)$  is in  $\mathcal{B}(V)^G$ .

The main problem of this paper is the construction of *invariant* hyperfunction solutions to the linear differential equation (2). In particular, when  $v(x)$  is a delta-function  $\delta(x)$  on  $V$ , this is a problem of the existence and the

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construction of  $G$ -invariant fundamental solution for  $P(x, \partial)$ . However, it is difficult to solve these problems for all  $G$ -invariant differential operators  $P(x, \partial)$  on  $V$ . In this paper, we assume that all the *homogeneous degrees* of the monomial components of  $P(x, \partial)$  are equal to a certain integer  $k$ . Then we say that  $P(x, \partial)$  is *homogeneous* and call the integer  $k$  the *total degree* of  $P(x, \partial)$ . Furthermore, we assume that the  $G$ -invariant hyperfunction  $v(x)$  is annihilated by a homogeneous  $G$ -invariant differential operator. Then we can prove that the solutions to (2) are expressed in terms of the Laurent expansion coefficients of the complex powers of the determinant functions. Thus we can apply the author's result in Muro [12].

We explain the organization of this paper. In §1, we describe the problem in a general setting and give some notions and notations we use in this paper. The important notions are *homogeneous* differential operators and *quasi-homogeneous* hyperfunctions. In §2, we introduce  $G$ -invariant differential equations on the real symmetric matrix space  $\text{Sym}_n(\mathbb{R})$  and hyperfunctions  $P^{[\vec{a}, s]}(x)$  given as linear combinations of complex powers of the determinant function on  $\text{Sym}_n(\mathbb{R})$ . A main result of this section is Proposition 2.1, that gives generators of the algebra of  $G$ -invariant differential operators. In §3, we define  $b_P$ -function that will play an important role in this paper and clarify its properties. In §4, we prove the first main theorem (Theorem 4.1), which shows that every  $G$ -invariant solution to  $P(x, \partial)u(x) = 0$  is given as a linear combination of quasi-homogeneous hyperfunctions under suitable conditions. In §5, we examine the properties of the complex powers  $P^{[\vec{a}, s]}(x)$  more precisely and, especially prove that every  $G$ -invariant quasi-homogeneous hyperfunction is given by a linear combination of Laurent expansion coefficients of  $P^{[\vec{a}, s]}(x)$  at on point  $s = \lambda$  and the converse is true. In §6, by applying the results in §5, we prove that there exists a  $G$ -invariant solution  $u(x)$  of  $P(x, \partial)u(x) = v(x)$  for a  $G$ -invariant quasi-homogeneous  $v(x)$  and that it is determined only by its  $b_P$ -function. In §6, we give a method to determine the order of pole of  $P^{[\vec{a}, s]}(x)$  as an application of the author's result in [12], and introduce "standard basis". It will be used in the algorithms in the later sections. In §8 and §9, we give some algorithms to construct  $G$ -invariant solutions for  $P(x, \partial)u(x) = 0$  and  $P(x, \partial)u(x) = v(x)$ , and in §10 we give some examples.

*The aim of this paper is not only to give solution spaces in an abstract form but also to write algorithms to construct all the solutions for given differential equations  $P(x, \partial)u(x) = 0$  or  $P(x, \partial)u(x) = v(x)$  using the Laurent expansion coefficients of the complex power function  $|\det(x)|^s$  ( $s \in \mathbb{C}$ ). In order to accomplish our purpose, we prove Theorem 4.1 in §4, Corollary 5.7 in §5, Theorem 6.1, Theorem 6.2 and Corollary 6.3 in §6, which are main theoretical results of this paper. They guarantee that every  $G$ -invariant hyperfunction solution for  $P(x, \partial)u(x) = 0$  or  $P(x, \partial)u(x) = v(x)$  can be written as a finite sum of the Laurent expansion coefficients of  $|\det(x)|^s$  and that the solution space is determined by the  $b_P$ -function of  $P(x, \partial)$  (see*

Definition 3.1). Then, we give algorithms to construct  $G$ -invariant hyperfunction solutions in §8 and §9 for given  $G$ -invariant differential equations and we give some examples in §10 for typical  $G$ -invariant differential equations.

The author want to stress that the algorithms (Algorithm 8.1, Algorithm 8.3 and Algorithm 8.2 in §8 and Algorithm 9.1 in §9) and the examples in §10 are important results of this paper as well as the main theorems (Theorem 4.1 in §4 and Theorem 6.1, Theorem 6.2, Corollary 6.3 in §6). For example, we prove in Proposition 10.2 that every  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solutions for the differential equation  $\det(x)u(x) = 0$  on  $V = \mathrm{Sym}_n(\mathbb{R})$  are linear sums of  $\mathrm{SL}_n(\mathbb{R})$ -invariant measures on the  $\mathrm{SL}_n(\mathbb{R})$ -orbits in the set  $S := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \det(x) = 0\}$  as an application of the algorithm. This is a natural extension of the fact that the hyperfunction solution to the differential equation  $xu(x) = 0$  on the real line  $x \in \mathbb{R}$  is only a constant multiple of the delta function  $u(x) = c \cdot \delta(x)$ .

P.-D. Methée's papers [6], [7] and [8] are pioneer works on this area. He solved the problem in the case that the indefinite rotation group acts on the real vector space. The problem of "construction of invariant hyperfunction solutions for invariant differential operators" seems to have been first considered by P.-D. Methée[6] in the framework of Schwartz's distribution theory. The book by N.N. Bogoliubov et al [1] on quantum field theory took up his works in the first chapter and present his results precisely. However Methée's method was rather primitive and it seems to be difficult to apply his method to the other cases. The author would like to propose more generally applicable method using holonomic system theory of  $D$ -modules in this paper. The author thinks that the method employed in this paper is more universal and applicable to the wide range of the actions of Lie groups to real vector spaces.

*Notations:* In this paper, for a square matrix  $x$ , we denote by  ${}^t x$ ,  $\mathrm{tr}(x)$  and  $\det(x)$  the transpose of  $x$ , the trace of  $x$  and the determinant of  $x$ , respectively. The complex numbers, the real numbers and the integers are denoted by  $\mathbb{C}$ ,  $\mathbb{R}$  and  $\mathbb{Z}$ , respectively. The subscripts signify the properties of the sets. For example,  $\mathbb{Z}_{\geq 0}$  means the non-negative integers and  $\mathbb{Z}_{> 0}$  means the positive integers.

## 1. FUNDAMENTAL DEFINITIONS AND PROBLEMS.

In this section we explain some definitions we shall use in this paper and describe the problem at a general setting.

Let  $V$  be a finite dimensional real vector space of dimension  $m$  with a linear coordinate  $(x_1, \dots, x_m)$ . Then a polynomial with complex coefficients on  $V$  is given as a complex finite linear combination of monomials  $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$  with  $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$ . We denote by  $\partial_i$  the partial derivative  $\frac{\partial}{\partial x_i}$  with respect to the variable  $x_i$ . We define a monomial of  $\frac{\partial}{\partial x_i}$ 's

by  $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}$  with  $\beta := (\beta_1, \dots, \beta_m) \in \mathbb{Z}_{\geq 0}^m$ . We define the degrees of multi-index by  $|\alpha| := \alpha_1 + \cdots + \alpha_m$  and  $|\beta| := \beta_1 + \cdots + \beta_m$ .

The generators  $x_1, \dots, x_m$  and  $\partial_1, \dots, \partial_m$  are commutative, respectively, and hence their algebras are polynomial algebras  $\mathbb{C}[x_1, \dots, x_m]$  and  $\mathbb{C}[\partial_1, \dots, \partial_m]$ , respectively. However,  $x_i$  and  $\partial_j$  are not commutative in general. They have a commutation relation

$$\partial_j x_i = x_i \partial_j + \delta_{ij} \quad (3)$$

where  $\delta_{ij}$  is the Kronecker's delta. The  $\mathbb{C}$ -algebra generated by  $x_1, \dots, x_m$  and  $\partial_1, \dots, \partial_m$  with the commutation relations (3) is a non-commutative  $\mathbb{C}$ -algebra. We denote it by  $D(V)$  and call an element of  $D(V)$  a *differential operator* on  $V$ . A differential operator on  $V$  is uniquely expressed as a finite linear combination of *monomial* differential operators

$$a_{\alpha\beta} x^\alpha \partial^\beta := a_{\alpha\beta} (x_1^{\alpha_1} \cdots x_m^{\alpha_m}) (\partial_1^{\beta_1} \cdots \partial_m^{\beta_m}) \quad (4)$$

with  $a_{\alpha\beta} \in \mathbb{C}$ . We call the expression of a differential operator using the monomial forms (4) a *normal form* of the differential operator.

We shall give definitions of a homogeneous differential operator in  $D(V)$  and its homogeneous degree.

**Definition 1.1** (homogeneous differential operators). For a given monomial differential operator  $a_{\alpha\beta} x^\alpha \partial^\beta$ , we call  $|\alpha| - |\beta|$  (resp.  $|\beta|$ ) a *homogeneous degree* (resp. an *order*) of the monomial differential operator  $a_{\alpha\beta} x^\alpha \partial^\beta$ . A *homogeneous differential operator of homogeneous degree  $k$*  in  $D(V)$  is a differential operator given as a finite linear combination of monomial differential operators of homogeneous degree  $k$ .

Let  $P(x, \partial)$  be a differential operator in  $D(V)$ . Then  $P(x, \partial)$  is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta. \quad (5)$$

Then each term

$$P_k(x, \partial) := \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta$$

is a homogeneous differential operator of degree  $k$ . Thus we see that

$$D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V)$$

where  $D_k(V)$  is a  $\mathbb{C}$ -vector subspace in  $D(V)$ . Note that  $D_k(V)$  is invariant under the linear coordinate transformation of  $V$  and a linear coordinate transformation of  $V$  gives a  $\mathbb{C}$ -algebra isomorphism of  $D(V)$  that preserves each  $D_k(V)$ .

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On the other hand,  $P(x, \partial)$  is expressed as

$$P(x, \partial) := \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta. \quad (6)$$

We call the *order* of  $P(x, \partial)$  the highest number  $k$  in the sum (6). Let  $q$  be the order of  $P(x, \partial)$ . Then the differential operator

$$\sigma(P)(x, \partial) := \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta| = q}} a_{\alpha\beta} x^\alpha \partial^\beta \quad (7)$$

is called the *principal part* of  $P(x, \partial)$  and the polynomial

$$\sigma(P)(x, \xi) := \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta| = q}} a_{\alpha\beta} x^\alpha \xi^\beta \quad (8)$$

is called the *principal symbol* of  $P(x, \partial)$ . Here  $\xi$  is the coordinate of the dual space of  $V$  corresponding to  $\partial$ .

From the definition,  $D_k(V)$  is closed under the additive operation, but not closed under the multiplicative operation. However we can easily check that

$$(a_{\alpha\beta} x^\alpha \partial^\beta) \cdot (b_{\gamma\delta} x^\gamma \partial^\delta) = \sum_{|\mu| + |\nu| = r} c_{\mu\nu} x^\mu \partial^\nu \quad (9)$$

where  $r = |\alpha| - |\beta| + |\gamma| - |\delta|$  and  $c_{\mu\nu} \in \mathbb{C}$  are zero except for a finite number of them. Namely we have

$$D_k(V) \times D_l(V) \ni (P, Q) \mapsto P \cdot Q \in D_{k+l}(V) \quad (10)$$

and  $\bigoplus_{k \in \mathbb{Z}} D_k(V)$  gives a gradation of  $D(V)$ .

Next we shall consider the differential operators invariant under the action of a subgroup  $G \subset \mathrm{GL}(V)$ , where  $\mathrm{GL}(V)$  is the general linear group on the vector space  $V$ . The action of  $g \in G$  to  $V$  leads to an algebra automorphism on  $D(V)$  since  $g \in G$  gives a linear coordinate transformation on  $V$ . We say that a differential operator invariant under the action of all  $g \in G$  a *G-invariant differential operator* on  $V$ . We denote  $D(V)^G$  the totality of  $G$ -invariant differential operators on  $V$ . We can easily check that  $D(V)^G$  a subalgebra of  $D(V)$  and  $D(V)^G = \bigoplus_{k \in \mathbb{Z}} D_k(V)^G := \bigoplus_{k \in \mathbb{Z}} D_k(V) \cap D(V)^G$  gives a natural gradation induced from the gradation  $D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V)$ .

*Remark 1.1.* Let  $P(x, \partial) \in D(V)$  be a homogeneous differential operator of degree  $k$  and let  $Q(x)$  be a homogeneous polynomial of degree  $l$ . Then the polynomial  $P(x, \partial)Q(x)$  is a homogeneous polynomial of degree  $k + l$ . Namely, the gradation  $D(V) = \bigoplus_{k \in \mathbb{Z}} D_k(V)$  is consistent with the gradation on the polynomial algebra by the homogeneous degree. Similarly we see that the gradation  $D(V)^G = \bigoplus_{k \in \mathbb{Z}} D_k(V)^G$  is consistent with the

gradation on the algebra of  $G$ -invariant polynomials by the homogeneous degree.

Let  $\mathcal{B}(V)$  be the space of hyperfunctions on  $V$  and let  $\mathcal{B}(V)^G$  be the space of  $G$ -invariant hyperfunctions on  $V$ . One of the important notions of this paper is  $G$ -invariant of quasi-homogeneous hyperfunctions.

**Definition 1.2** (quasi-homogeneous hyperfunctions). We say that  $v(x) \in \mathcal{B}(V)$  is *quasi-homogeneous* if and only if there exist a complex number  $\lambda \in \mathbb{C}$  and a non-negative integer  $k \in \mathbb{Z}_{\geq 0}$  satisfying

$$\underbrace{F_{r,\lambda} \circ F_{r,\lambda} \circ \cdots \circ F_{r,\lambda}}_{k+1}(v) = 0 \quad (11)$$

for all  $r \in \mathbb{R}_{>0}$  where  $F_{r,\lambda}(v) := v(r \cdot x) - r^\lambda v(x)$ . We call  $\lambda \in \mathbb{C}$  the *homogeneous degree* (or simply *degree*) of  $v(x)$  and  $k \in \mathbb{Z}_{\geq 0}$  the *quasi-degree* of  $v(x)$ . It is easily checked that (11) is equivalent to

$$(\vartheta - \lambda)^{k+1} v(x) = 0 \quad (12)$$

with  $\vartheta := \sum_{i=1}^m x_i \partial_i$ . In particular, when a quasi-homogeneous function  $v(x)$  is of quasi-degree  $k$  and not  $k-1$ , we say that  $v(x)$  is quasi-homogeneous of *proper quasi-degree*  $k$ .

For example, let  $P(x)$  be a homogeneous polynomial of degree  $n$  and let  $\lambda$  be a complex number with sufficiently large real part. Then  $|P(x)|^\lambda$  is a quasi-homogeneous hyperfunction of degree  $\lambda n$  and quasi-degree 0. More generally,  $|P(x)|^\lambda (\log |P(x)|)^k$  is a quasi-homogeneous hyperfunction of degree  $\lambda n$  and quasi-degree  $k$ .

We use the following notations in this paper.

1.  $QH(\lambda) := \{u(x) \in \mathcal{B}(V) \mid u(x) \text{ is quasi-homogeneous of degree } \lambda \in \mathbb{C}\}.$
2.  $QH(\lambda)^G := QH(\lambda) \cap \mathcal{B}(V)^G.$
3.  $QH := \bigoplus_{\lambda \in \mathbb{C}} QH(\lambda).$
4.  $QH^G := \bigoplus_{\lambda \in \mathbb{C}} QH(\lambda)^G.$

**Proposition 1.1.** *Let  $P(x, \partial) \in D(V)$  (resp.  $\in D(V)^G$ ) be a non-zero homogeneous differential operator of homogeneous degree  $\mu$ . If  $f(x) \in \mathcal{B}(V)$  (resp.  $\in \mathcal{B}(V)^G$ ) is quasi-homogeneous of degree  $\lambda \in \mathbb{C}$ , then  $P(x, \partial)f(x) \in \mathcal{B}(V)$  (resp.  $\in \mathcal{B}(V)^G$ ) is quasi-homogeneous of degree  $\lambda + \mu \in \mathbb{C}$ .*

*Proof.* Let  $P(x, \partial) = \sum_{|\alpha| - |\beta| = \mu} a_{\alpha\beta} x^\alpha \partial^\beta \in D(V)$  be a homogeneous differential operator of degree  $\mu$  and let  $\vartheta := \sum_{i=1}^m x_i \partial_i$ . We prove that

$$P(x, \partial)(\vartheta - \lambda) = (\vartheta - \lambda - \mu)P(x, \partial). \quad (13)$$

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For a monomial term  $a_{\alpha\beta}x^\alpha\partial^\beta$  in  $P(x, \partial)$ , we have

$$\begin{aligned} a_{\alpha\beta}x^\alpha\partial^\beta(\vartheta - \lambda) &= a_{\alpha\beta}x^\alpha(\vartheta - \lambda + |\beta|)\partial^\beta \\ &= a_{\alpha\beta}(\vartheta - \lambda + |\beta| - |\alpha|)x^\alpha\partial^\beta \\ &= (\vartheta - \lambda + |\beta| - |\alpha|)a_{\alpha\beta}x^\alpha\partial^\beta = (\vartheta - \lambda - \mu)a_{\alpha\beta}x^\alpha\partial^\beta, \end{aligned}$$

and hence we have (13). Thus for a quasi-homogeneous  $f(x) \in \mathcal{B}(V)$  of degree  $\lambda$ , we have

$$(\vartheta - \lambda - \mu)^k P(x, \partial)f(x) = P(x, \partial)(\vartheta - \lambda)^k f(x) = 0$$

for some  $k \in \mathbb{Z}_{>0}$ . Then we see that  $P(x, \partial)f(x)$  is a quasi-homogeneous hyperfunction of degree  $\lambda + \mu$ .

For  $P(x, \partial) \in D(V)^G$  and  $f(x) \in \mathcal{B}(V)^G$ , we can prove it in the same way.  $\square$

*Remark 1.2.* The notion of *quasi-homogeneous* hyperfunctions is the same as that of *associated homogeneous* generalized functions introduced by I.M. Gelfand and G.E. Shilov [3], Chapter 1, §4 when we consider the functions of one variable. In other words, as far as we only consider the case of one-variable function, “associated homogeneous generalized functions of order  $k$  and of degree  $\lambda$ ” defined in the Gelfand-Shilov’s book is just the same as “quasi-homogeneous hyperfunctions of degree  $\lambda$  and of quasi-degree  $k$ ” defined in this paper. Gelfand and Shilov introduced this notion to characterize Laurent expansion coefficients of the complex power  $x^s$  of homogeneous function  $x$  with respect to the complex variable  $s \in \mathbb{C}$ . We see later (in §5) that  $G$ -invariant quasi-homogeneous hyperfunctions are obtained as Laurent expansion coefficients of the complex powers  $|P(x)|_i^s$  of  $G$ -invariant polynomial  $P(x)$  with respect to the complex variable  $s \in \mathbb{C}$  in the case of  $V = \text{Sym}_n(\mathbb{R})$  and  $G = \text{SL}_n(\mathbb{R})$ .

Now we complete the preparation to explain our problem in general situation. The problems we shall propose in this paper are the following ones.

**Problem 1.1** (Main Problems). Let  $P(x, \partial) \in D(V)^G$  be a given  $G$ -invariant *homogeneous* differential operator.

1. Construct a basis of  $G$ -invariant hyperfunction solutions  $u(x) \in \mathcal{B}(V)^G$  to the differential equation

$$P(x, \partial)u(x) = 0.$$

2. Construct a  $G$ -invariant hyperfunction solution  $u(x) \in \mathcal{B}(V)^G$  to the differential equation

$$P(x, \partial)u(x) = v(x).$$



for a given *quasi-homogeneous* hyperfunction  $v(x) \in \mathcal{B}(V)^G$ . In particular, when  $v(x) = \delta(x)$ , it is a problem to find a  $G$ -invariant fundamental solution.

In this paper, we give a method to construct solutions to the problems in Problem 1.1 in the case that  $V := \text{Sym}_n(\mathbb{R})$  and  $G := \text{SL}_n(\mathbb{R})$  and construct solutions actually in some typical examples. The condition that  $v(x)$  is quasi-homogeneous in the second problem of Problem 1.1 may seem to be highly restrictive at first glance. However, in our case, we see that many important  $G$ -invariant hyperfunctions such as singular invariant hyperfunctions (like  $\delta(x)$ ) are contained in this class, so the author thinks that this is a class wide enough for our problem.

## 2. COMPLEX POWERS OF DETERMINANT FUNCTIONS AND INVARIANT DIFFERENTIAL OPERATORS ON THE SYMMETRIC MATRIX SPACE.

From now on, we shall deal with the symmetric matrix space  $\text{Sym}_n(\mathbb{R})$  on which the special linear group  $\text{SL}_n(\mathbb{R})$  acts naturally. Let  $V := \text{Sym}_n(\mathbb{R})$  be the space of  $n \times n$  symmetric matrices over the real field  $\mathbb{R}$  and let  $\text{SL}_n(\mathbb{R})$  be the special linear group over  $\mathbb{R}$  of degree  $n$ . Then the group  $G := \text{SL}_n(\mathbb{R})$  acts on the vector space  $V$  by the representation

$$\rho(g) : x \mapsto g \cdot x := gx^t g,$$

with  $x \in V$  and  $g \in G$ . The pair  $(G, V) = (\text{SL}_n(\mathbb{R}), \text{Sym}_n(\mathbb{R}))$  is the object that we shall study in this paper.

The vector space  $V$  decomposes into a finite number of  $\text{GL}_n(\mathbb{R})$ -orbits;

$$V := \bigsqcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n-i}} S_i^j \quad (14)$$

where

$$S_i^j := \{x \in \text{Sym}_n(\mathbb{R}) \mid \text{sgn}(x) = (j, n-i-j)\} \quad (15)$$

with integers  $0 \leq i \leq n$  and  $0 \leq j \leq n-i$ . In particular, an orbit in  $S$  is a  $G$ -orbit. A  $G$ -orbit in  $S$  is called a *singular orbit*. The subset  $S_i := \{x \in V \mid \text{rank}(x) = n-i\}$  is the set of elements of rank  $n-i$ . It is easily seen that  $S := \bigsqcup_{1 \leq i \leq n} S_i$  and  $S_i = \bigsqcup_{0 \leq j \leq n-i} S_i^j$ . The strata  $\{S_i^j\}_{1 \leq i \leq n, 0 \leq j \leq n-i}$  have the following closure inclusion relation

$$\overline{S_i^j} \supset S_{i+1}^{j-1} \cup S_{i+1}^j, \quad (16)$$

where  $\overline{S_i^j}$  means the *closure* of the stratum  $S_i^j$ .

We denote  $P(x) := \det(x)$  and we set  $S := \{x \in V \mid \det(x) = 0\}$ . The subset  $V - S$  decomposes into  $n+1$  connected components,

$$V_i := \{x \in \text{Sym}_n(\mathbb{R}) \mid \text{sgn}(x) = (i, n-i)\} \quad (17)$$

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with  $i = 0, 1, \dots, n$ . Here,  $\text{sgn}(x)$  for  $x \in \text{Sym}_n(\mathbb{R})$  is the signature of the quadratic form  $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$  on  $\vec{v} \in \mathbb{R}^n$ . We define the complex power function of  $P(x)$  by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in V_i, \\ 0 & \text{if } x \notin V_i. \end{cases} \quad (18)$$

for a complex number  $s \in \mathbb{C}$ . These functions are well defined on  $V - S$  but it is not clear whether they are extended to the whole space  $V$ . In order to make  $|P(x)|_i^s$  well defined as a hyperfunction on  $V$ , we use the analytic continuation with respect to  $s \in \mathbb{C}$ . Let  $\mathcal{S}(V)$  be the space of rapidly decreasing smooth functions on  $V$ . For  $f(x) \in \mathcal{S}(V)$ , the integral

$$Z_i(f, s) := \int_V |P(x)|_i^s f(x) dx, \quad (19)$$

is convergent if the real part  $\Re(s)$  of  $s$  is sufficiently large and is meromorphically extended to the whole complex plane. Thus we can regard  $|P(x)|_i^s$  as a tempered distribution — and hence a hyperfunction — with a meromorphic parameter  $s \in \mathbb{C}$ . We consider a linear combination of the hyperfunctions  $|P(x)|_i^s$

$$P^{[\vec{a}, s]}(x) := \sum_{i=0}^n a_i \cdot |P(x)|_i^s \quad (20)$$

with  $s \in \mathbb{C}$  and  $\vec{a} := (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$ . Then  $P^{[\vec{a}, s]}(x)$  is a hyperfunction with a meromorphic parameter  $s \in \mathbb{C}$ , and depends on  $\vec{a} \in \mathbb{C}^{n+1}$  linearly.

*Remark 2.1.* We call  $S := \{x \in V; \det(x) = 0\}$  a *singular set* of  $V$  and we say that a hyperfunction  $f(x)$  on  $V$  is *singular* if the support of  $f(x)$  is contained in the singular set  $S$ . In particular, any singular invariant hyperfunction is written as a finite sum of quasi-homogeneous hyperfunctions. In addition, if  $f(x)$  is  $\text{SL}_n(\mathbb{R})$ -invariant, i.e.,  $f(g \cdot x) = f(x)$  for all  $g \in \text{SL}_n(\mathbb{R})$ , we call  $f(x)$  a *singular invariant* hyperfunction on  $V$ . Any negative-order coefficient of a Laurent expansion of  $P^{[\vec{a}, s]}(x)$  is a singular invariant hyperfunction, since the integral

$$\int f(x) P^{[\vec{a}, s]}(x) dx = \sum_{i=0}^n Z_i(f, s) \quad (21)$$

is an entire function with respect to  $s \in \mathbb{C}$  if  $f(x) \in C_0^\infty(V - S)$ , where  $C_0^\infty(V - S)$  is the space of compactly supported  $C^\infty$ -functions on  $V - S$ . Conversely, we have the following proposition. Any singular  $G$ -invariant hyperfunction on  $V$  is given as a linear combination of some negative-order coefficients of Laurent expansions of  $P^{[\vec{a}, s]}(x)$  at various poles and for some  $\vec{a} \in \mathbb{C}^{n+1}$ . See [10] and [11]. Thus we see that any singular invariant hyperfunction is written as a linear combination of quasi-homogeneous hyperfunctions.

As defined in Definition 1.1, homogeneous differential operator of degree  $k \in \mathbb{Z}$  is given by

$$P(x, \partial) = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^m \\ |\alpha| - |\beta| = k}} a_{\alpha\beta} x^\alpha \partial^\beta$$

where  $m = n(n+1)/2$  in the case of symmetric matrix space. The notations here are written as

$$x = (x_{ij})_{n \geq j \geq i \geq 1}, \quad \partial = (\partial_{ij}) = \left( \frac{\partial}{\partial x_{ij}} \right)_{n \geq j \geq i \geq 1}$$

$$x^\alpha = \prod_{n \geq j \geq i \geq 1} x_{ij}^{\alpha_{ij}}, \quad \partial^\beta = \prod_{n \geq j \geq i \geq 1} \partial_{ij}^{\beta_{ij}}$$

with

$$\alpha = (\alpha_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |\alpha| = \sum_{n \geq j \geq i \geq 1} \alpha_{ij}$$

and

$$\beta = (\beta_{ij}) \in \mathbb{Z}_{\geq 0}^m, \quad |\beta| = \sum_{n \geq j \geq i \geq 1} \beta_{ij}.$$

We define  $\partial^*$  by

$$\partial^* = (\partial_{ij}^*) = \left( \epsilon_{ij} \frac{\partial}{\partial x_{ij}} \right), \text{ and } \epsilon_{ij} := \begin{cases} 1 & i = j \\ 1/2 & i \neq j \end{cases} \quad (22)$$

We shall give some examples of  $G$ -invariant homogeneous differential operators.

**Example 2.1.** We give here *fundamental* invariant homogeneous differential operators in the sense that they form a complete set of generators of  $D(V)^{\mathbf{SL}_n(\mathbb{R})}$  and  $D(V)^{\mathbf{GL}_n(\mathbb{R})}$ , which we shall prove in Proposition 2.1.

1. Let  $h$  and  $n$  be positive integers with  $1 \leq h \leq n$ . A sequence of increasing integers  $p = (p_1, \dots, p_h) \in \mathbb{Z}^h$  is called an *increasing sequence in  $[1, n]$  of length  $h$*  if it satisfies  $1 \leq p_1 < \dots < p_h \leq n$ . We denote by  $\text{IncSeq}(h, n)$  the set of increasing sequences in  $[1, n]$  of length  $h$ .
2. For two sequences  $p = (p_1, \dots, p_h)$  and  $q = (q_1, \dots, q_h) \in \text{IncSeq}(h, n)$  and for an  $n \times n$  symmetric matrix  $x = (x_{ij}) \in \text{Sym}_n(\mathbb{R})$ , we define an  $h \times h$  matrix  $x_{(p,q)}$  by

$$x_{(p,q)} := (x_{p_i, q_j})_{1 \leq i \leq j \leq h}.$$

In the same way, for an  $n \times n$  symmetric matrix  $\partial = (\partial_{ij})$  of differential operators, we define an  $h \times h$  matrix  $\partial_{(p,q)}$  of differential operators by

$$\partial_{(p,q)}^* := (\partial_{p_i, q_j}^*)_{1 \leq i \leq j \leq h}.$$

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3. For an integer  $h$  with  $1 \leq h \leq n$ , we define

$$P_h(x, \partial) := \sum_{p, q \in \text{IncSeq}(h, n)} \det(x_{(p, q)}) \det(\partial_{(p, q)}^*). \quad (23)$$

4. In particular,  $P_n(x, \partial) = \det(x) \det(\partial^*)$  and *Euler's differential operator* is given by

$$P_1(x, \partial) = \sum_{n \geq j \geq i \geq 1} x_{ij} \frac{\partial}{\partial x_{ij}} = \text{tr}(x \cdot \partial^*). \quad (24)$$

These are all homogeneous differential operators of degree 0 and invariant under the action of  $\mathbf{GL}_n(\mathbb{R})$ , and hence it is also invariant under the action of  $\mathbf{G} := \mathbf{SL}_n(\mathbb{R}) \subset \mathbf{GL}_n(\mathbb{R})$ .

5.  $\det(x)$  and  $\det(\partial^*)$  are homogeneous differential operators of degree  $n$  and  $-n$ , respectively. They are invariant under the action of  $\mathbf{G} := \mathbf{SL}_n(\mathbb{R})$ , and relatively invariant differential operators under the action of  $\mathbf{GL}_n(\mathbb{R})$ , with characters  $\chi(g) := \det(g)^2$  and  $\chi^{-1}(g) := \det(g)^{-2}$ , respectively.

**Proposition 2.1.**

1. Every  $\mathbf{GL}_n(\mathbb{R})$ -invariant differential operator in  $D(V)$  can be expressed as a polynomial in  $P_i(x, \partial)$  ( $i = 1, \dots, n$ ) defined in (23). The algebra  $D(V)^{\mathbf{GL}_n(\mathbb{R})}$  is isomorphic to the polynomial algebra  $\mathbb{C}[P_1, \dots, P_n]$ .
2. Every  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator in  $D(V)$  can be expressed as a polynomial in  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  (see Remark 2.2). The algebra  $D(V)^{\mathbf{SL}_n(\mathbb{R})}$  is generated by  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  but is not isomorphic to the polynomial algebra.

*Remark 2.2.* The differential operators  $\det(x)$  and  $\det(\partial^*)$  are not commutative. Then the polynomial expression of an  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial)$  in terms of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  is not unique. In this paper, by “polynomial” expression of  $P(x, \partial)$  in terms of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$ , we mean an expression as a finite sum of monomial terms of the form

$$P_1(x, \partial)^{h_1} \cdots P_{n-1}(x, \partial)^{h_{n-1}} (\det(x))^{h_n} (\det(\partial^*))^{h_{n+1}}$$

with non-negative integers  $h_i$  ( $i = 1, \dots, n+1$ ).

*Proof.* The proof of Proposition 2.1-1 is given in H. Maass [5] pp.66-67. We go to the proof of Proposition 2.1-2.

Let  $Q(x, \partial)$  be an  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator in  $D(V)$ . We want to prove that  $Q(x, \partial)$  can be expressed as a polynomial in  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$ . We first show that it is sufficient to

prove it when  $Q(x, \partial)$  is a homogeneous differential operator. Indeed, any  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator  $Q(x, \partial)$  can be decomposed as

$$Q(x, \partial) = \sum_{k \in \mathbb{Z}} Q^{(k)}(x, \partial)$$

where  $Q^{(k)}(x, \partial)$  is the homogeneous part of degree  $k$ , i.e., the sum of all the monomial terms of degree  $k$ . Let  $c \in \mathbb{R}$  and  $g \in \mathbf{SL}_n(\mathbb{R})$ . Then we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} c^k Q^{(k)}(x, \partial) &= \sum_{k \in \mathbb{Z}} Q^{(k)}(c \cdot x, c^{-1} \cdot \partial) \\ &= Q(c \cdot x, c^{-1} \cdot \partial) = Q(c \cdot g \cdot x, c^{-1} \cdot {}^t g^{-1} \cdot \partial) \\ &= \sum_{k \in \mathbb{Z}} Q^{(k)}(c \cdot g \cdot x, c^{-1} \cdot {}^t g^{-1} \partial) = \sum_{k \in \mathbb{Z}} c^k Q^{(k)}(g \cdot x, {}^t g^{-1} \partial), \end{aligned}$$

and hence we have

$$Q^{(k)}(x, \partial) = Q^{(k)}(g \cdot x, {}^t g^{-1} \partial),$$

for each  $k \in \mathbb{Z}$ . This means that each  $Q^{(k)}(x, \partial)$  is  $\mathbf{SL}_n(\mathbb{R})$ -invariant. Then if we prove that  $Q(x, \partial)$  can be expressed as a polynomial in  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  when  $Q(x, \partial)$  is a homogeneous  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator, then it is valid for any  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator.

Now we suppose that  $Q(x, \partial)$  is a homogeneous  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator of degree  $k \in \mathbb{Z}$ . If  $k = 0$ , then  $Q(x, \partial)$  is  $\mathbf{GL}_n(\mathbb{R})$ -invariant, and hence we have proved it by Proposition 2.1-1. Then we suppose that  $k \neq 0$ . Since  $Q(x, \partial)$  is homogeneous and  $\mathbf{SL}_n(\mathbb{R})$ -invariant,  $Q(x, \partial)$  is relatively invariant under the action of  $\mathbf{GL}_n(\mathbb{R})$ , and hence we have

$$Q(g \cdot x, {}^t g^{-1} \cdot \partial) = \det(g)^{2k'} Q(x, \partial) \quad (25)$$

for all  $g \in \mathbf{GL}_n(\mathbb{R})$  with  $k' = k/n \in \mathbb{Z} - \{0\}$ .

In fact, since  $Q(x, \partial)$  is relatively invariant under the action of  $\mathbf{GL}_n(\mathbb{R})$ , there exists  $r \in \mathbb{Z}$  satisfying

$$Q(g \cdot x, {}^t g^{-1} \cdot \partial) = \det(g)^r Q(x, \partial)$$

for all  $g \in \mathbf{GL}_n(\mathbb{R})$ . We shall prove that  $r$  is an even integer. Since  $Q(x, \xi)$  is a non-zero polynomial on  $V \times V^*$ . There exists a suitable point  $(x_0, \xi_0) \in V \times V^*$  such that  $Q(x_0, \xi_0) \neq 0$ . In particular, we may take  $x_0$  to be positive definite. By moving the point  $(x_0, \xi_0)$  by the action of  $\mathbf{GL}_n(\mathbb{R})$ , we may assume that  $x_0$  and  $\xi_0$  have the forms

$$x_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad \xi_0 = \begin{bmatrix} y_1 & 0 & \cdots & 0 & 0 \\ 0 & y_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & y_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & y_n \end{bmatrix}.$$

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If  $r$  is odd, then by taking  $g = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$ , we have  $\det(g) = -1$ .

Then we have

$$\begin{aligned} Q(x_0, \xi_0) &= Q(g \cdot x_0, {}^t g^{-1} \cdot \xi_0) = \det(g)^r Q(x_0, \xi_0) \\ &= (-1)^r Q(x_0, \xi_0) = (-1) Q(x_0, \xi_0). \end{aligned}$$

From the assumption that  $Q(x_0, \xi_0) \neq 0$ , this is a contradiction. Then we have  $r$  is an even integer. On the other hand, since  $Q(x, \partial)$  is homogeneous of degree  $k$ , the character  $\det(g)^r$  is a homogeneous rational function on  $\mathbf{GL}_n(\mathbb{R})$  of degree  $2k$ . Then we have  $2k = rn$ . Since  $r$  is even,  $k$  is divisible by  $n$  and  $r = 2(k/n) = 2k'$ . Thus we have (25).

We shall prove that  $Q(x, \partial)$  is expressed as a polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  if  $Q(x, \partial)$  is homogeneous of degree  $k \in \mathbb{Z} - \{0\}$  and  $\mathbf{SL}_n(\mathbb{R})$ -invariant in the following. We use the induction on the order of  $Q(x, \partial)$ .

Suppose that the order of  $Q(x, \partial)$  is zero. Then  $Q(x, \partial)$  is a polynomial in  $x$ . Since  $Q(x, \partial)$  is  $\mathbf{SL}_n(\mathbb{R})$ -invariant, it is expressed as a polynomial in  $\det(x)$ , and hence the proposition is valid.

Next we suppose that any  $Q(x, \partial)$  is expressed as an polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  if the order of  $Q(x, \partial)$  is less than  $q-1$  and if  $Q(x, \partial)$  is homogeneous of degree  $k \in \mathbb{Z} - \{0\}$  and  $\mathbf{SL}_n(\mathbb{R})$ -invariant. Then we take one  $Q(x, \partial)$  whose order is  $q$  and which is supposed to be homogeneous of degree  $k \in \mathbb{Z} - \{0\}$  and  $\mathbf{SL}_n(\mathbb{R})$ -invariant. Note that  $k$  is divisible by  $n$ . We put  $k' := k/n$  and

$$F(x, \partial) := \begin{cases} Q(x, \partial) \det(\partial)^{k'} & \text{if } k' > 0 \\ \det(x)^{-k'} Q(x, \partial) & \text{if } k' < 0 \end{cases}$$

Then  $F(x, \partial)$  is homogeneous of degree 0 and  $\mathbf{SL}_n(\mathbb{R})$ -invariant. Thus, by Proposition 2.1-1,  $F(x, \partial)$  is written as a polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$ . Therefore, the principal symbol  $\sigma(F)(x, \xi)$  is a polynomial of  $P_i(x, \xi)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\xi^*)$ . Here  $\xi$  is the dual coordinate corresponding to  $\partial$ . Then

$$\sigma(Q)(x, \xi) = \begin{cases} \sigma(F)(x, \xi) \det(\xi)^{-k'} & \text{if } k' > 0 \\ \det(x)^{k'} \sigma(F)(x, \xi) & \text{if } k' < 0 \end{cases}$$

is not only a rational function of  $P_i(x, \xi)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\xi^*)$  but also a polynomial of them since  $P_i(x, \xi)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\xi^*)$  are algebraically independent. Thus we can write

$$\sigma(Q)(x, \xi) = R(P_1(x, \xi), \dots, P_{n-1}(x, \xi), \det(x), \det(\xi^*))$$

where  $R$  is a polynomial. Then by putting

$$Q_1(x, \partial) := Q(x, \partial) - R(P_1(x, \partial), \dots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)),$$

the order of  $Q_1(x, \partial)$  is less than  $q - 1$  and  $Q_1(x, \partial)$  is homogeneous of degree  $k \in \mathbb{Z} - \{0\}$  and  $\mathrm{SL}_n(\mathbb{R})$ -invariant. Therefore, from the induction hypothesis,  $Q_1(x, \partial)$  is expressed as a polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n - 1$ ),  $\det(x)$  and  $\det(\partial^*)$  and so is

$$Q(x, \partial) = Q_1(x, \partial) - R(P_1(x, \partial), \dots, P_{n-1}(x, \partial), \det(x), \det(\partial^*)).$$

Thus, by induction of the order, we have proved that  $Q(x, \partial)$  is expressed as a polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n - 1$ ),  $\det(x)$  and  $\det(\partial^*)$  if  $Q(x, \partial)$  is homogeneous of degree  $k \in \mathbb{Z} - \{0\}$  and  $\mathrm{SL}_n(\mathbb{R})$ -invariant.  $\square$

### 3. $b_P$ -FUNCTIONS OF INVARIANT DIFFERENTIAL OPERATORS.

As we will see later (Theorem 4.1), the most important object for our problems is the  $b_P$ -function (Definition 3.1) of the invariant differential operator  $P(x, \partial)$  and its homogeneous degree. In this section we shall define  $b_P$ -functions and give some examples.

**Proposition 3.1.** *Let  $P(x, \partial) \in D(V)^G$  be a homogeneous differential operator.*

1. *The homogeneous degree of  $P(x, \partial)$  is in  $(n \cdot \mathbb{Z})$ . Namely the homogeneous degree is divisible by  $n$ . If the homogeneous degree of  $P(x, \partial)$  is  $nk$ , then it is relatively invariant under the action of  $g \in \mathrm{GL}_n(\mathbb{R})$  corresponding to the character  $\det(g)^{2k}$ , i.e.,*

$$P(g \cdot x, {}^t g^{-1} \cdot \partial) = \det(g)^{2k} P(x, \partial).$$

2. *If the homogeneous degree of  $P(x, \partial)$  is  $nk$  with  $k \in \mathbb{Z}$ , then we have*

$$P(x, \partial)(\det x)^s = b_P(s)(\det x)^{s+k} \quad (26)$$

where  $b_P(s)$  is a polynomial in  $s \in \mathbb{C}$  and  $x \in \mathrm{Sym}_n(\mathbb{R})$  is positive definite. We have also

$$\begin{aligned} P(x, \partial)P^{[\vec{a}, s]}(x) &= b_P(s) \det(x)^k P^{[\vec{a}, s]}(x) \\ &= b_P(s) \mathrm{sgn}(\det(x))^k P^{[\vec{a}, s+k]}(x) \\ &= b_P(s) P^{[\vec{a}^{\#k}, s+k]}(x) \end{aligned} \quad (27)$$

for all  $x \in V - S$ . Here we put

$$\vec{a}^{\#k} := ((-1)^{nk} a_0, (-1)^{(n-1)k} a_1, \dots, a_n) \in \mathbb{C}^{n+1}. \quad (28)$$

3. *If the homogeneous degree of  $P(x, \partial)$  is  $nk$  with  $k < 0$ , then we have  $b^{-k}(s-1)|b_P(s)$  where  $b^{-k}(s-1) := b(s-1)b(s-2) \cdots b(s-(-k))$  with  $b(s) := \prod_{i=1}^n (s + \frac{i+1}{2})$ .*

*Proof.* 1. By Proposition 2.1, any  $\mathrm{SL}_n(\mathbb{R})$ -invariant  $P(x, \partial)$  is written as a polynomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n - 1$ ),  $\det(x)$  and  $\det(\partial^*)$ . The homogeneous degrees of  $P_i(x, \partial)$  ( $i = 1, \dots, n - 1$ ) are 0 and those

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of  $\det(x)$  and  $\det(\partial^*)$  are  $n$  and  $-n$ , respectively. Therefore the homogeneous degree of  $P(x, \partial)$  is a multiple of  $n$ . On the other hand, the operators  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ) are absolutely invariant under the action of  $g \in \mathbf{GL}_n(\mathbb{R})$  and the operators  $\det(x)$  and  $\det(\partial^*)$  are relatively invariant under the action of  $g \in \mathbf{GL}_n(\mathbb{R})$  corresponding to the character  $\det(g)^2$  and  $\det(g)^{-2}$ , respectively. Then each monomial of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  in  $P(x, \partial)$  is relatively invariant and the corresponding character is determined by its homogeneous degree. Then, if  $P(x, \partial)$ 's homogeneous degree is  $nk$ , it is relatively invariant under the action of  $g \in \mathbf{GL}_n(\mathbb{R})$  corresponding to the character  $\det(g)^{2k}$ .

2. Note that  $P^{[\vec{a}, s]}(x) = \sum_{i=1}^n a_i |P(x)|_i^s$ . For  $x \in V_n$ ,  $x$  is positive definite matrix and  $|P(x)|_n^s = (\det(x))^s$ . Then there exists a polynomial  $b_P(s)$  satisfying

$$\begin{aligned} P(x, \partial)|P(x)|_n^s &= P(x, \partial)(\det(x))^s \\ &= b_P(s)(\det(x))^{s+k} \\ &= b_P(s)|P(x)|_n^{s+k} \end{aligned}$$

since  $P(x, \partial)|P(x)|_n^s$  is a relatively invariant function under the action of  $g \in \mathbf{GL}_n(\mathbb{R})$  corresponding to the character  $(\det(g))^{2(s+k)}$  and since  $V_n$  is a  $\mathbf{GL}_n(\mathbb{R})$ -orbit. Here, note that the equation

$$P(x, \partial)(\det(x))^s = b_P(s)(\det(x))^{s+k} \quad (29)$$

is extended to any  $x \in V - S$  by an analytic continuation through the complex domain  $V \otimes \mathbb{C}$ .

Next, for  $x \in V_i$ , we have

$$|P(x)|_i^s = |\det(x)|^s = ((-1)^{n-i}(\det(x)))^s = (-1)^{(n-i)s}(\det(x))^s. \quad (30)$$

However, note that the value of the complex power  $(-1)^{(n-i)s}$  is determined by taking a suitable branch of analytic continuation, but it must be compatible with the branch of analytic continuation of  $(\det(x))^s$ .

Then, for  $x \in V_i$ , we have

$$\begin{aligned} P(x, \partial)|P(x)|_i^s &= P(x, \partial)((-1)^{n-i}(\det(x)))^s \\ &= (-1)^{(n-i)s}P(x, \partial)(\det(x))^s \\ &= (-1)^{(n-i)s}b_P(s)(\det(x))^{s+k} \quad (\text{by (29)}) \\ &= (-1)^{(n-i)s}b_P(s)(-1)^{-(n-i)(s+k)}|P(x)|_i^{s+k} \quad (\text{by (30)}) \\ &= (-1)^{-(n-i)k}b_P(s)|P(x)|_i^{s+k} \\ &= (-1)^{(n-i)k}b_P(s)|P(x)|_i^{s+k}. \end{aligned}$$

Then we have

$$P(x, \partial)P^{[\vec{a}, s]}(x) = b_P(s)P^{[\vec{a}^{\#k}, s+k]}(x)$$

for all  $x \in V - S$ .



3. Let  $P(x, \partial)$  be a homogeneous  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator of degree  $nk$  with  $k < 0$ . From the result in Proposition 2.1-2, each monomial in  $P(x, \partial)$  has  $(\det(\partial^*))^r$  with  $r > (-k)$ . Namely, for a monomial in  $P(x, \partial)$

$$\prod_{h=1}^{n-1} P_h(x, \partial)^{p_h} (\det(x))^q (\det(\partial^*))^r \quad (31)$$

with  $p_h (h = 1, \dots, n-1), q, r \in \mathbb{Z}_{\geq 0}$ ,  $r$  must be greater than  $-k$ . Since

$$(\det(\partial^*))^r (\det(x))^s = b(s-1)b(s-2) \cdots b(s-r)(\det(x))^{s-r},$$

the  $b_P$ -function of  $P(x, \partial)$  must contain  $b^{-k}(s-1) := b(s-1)b(s-2) \cdots b(s-(-k))$  as a divisor.  $\square$

Now we can give the definition of  $b_P$ -function for a given  $\mathbf{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial)$ .

**Definition 3.1** ( $b_P$ -function). Let  $P(x, \partial) \in D(V)^G$  be a homogeneous differential operator of homogeneous degree  $k$ . We call  $b_P(s)$  in (26) the  $b_P$ -function of  $P(x, \partial)$ .

**Example 3.1.** The  $b_P$ -functions of the invariant differential operators given in Example 2.1 can be explicitly computed by using Capelli's identity.

1. Consider the invariant differential operators

$$P_h(x, \partial) := \sum_{p, q \in \text{IncSeq}(h, n)} \det(x_{(p, q)}) \det(\partial_{(p, q)}^*).$$

defined by (23) for  $h = 1, \dots, n$ . These are not only  $\mathbf{SL}_n(\mathbb{R})$ -invariant but also  $\mathbf{GL}_n(\mathbb{R})$ -invariant and their homogeneous degree is 0. The  $b_P$ -function of  $P_h(x, \partial)$  is given by

$$b_P(s) = c_h \cdot \prod_{i=1}^h \left(s + \frac{i-1}{2}\right) \quad (32)$$

with a non-zero constant  $c_h$ .

2. The  $b_P$ -function of  $P(x, \partial) := \det(\partial^*)$  is given by

$$b_P(s) = c_n \cdot \prod_{i=1}^n \left(s + \frac{i-1}{2}\right) \quad (33)$$

with a non-zero constant  $c_n$ .

3. The  $b_P$ -function of  $P(x, \partial) := \det(x)$  is given by

$$b_P(s) = 1. \quad (34)$$

The rationality and the negativity of the roots of the  $b_P$ -function for  $P(x, \partial) := \det(\partial^*)$  is a consequence of the rationality theorem of  $b$ -function by Kashiwara[4]. However the  $b_P$ -function for a homogeneous differential

operator  $P(x, \partial)$  in this paper is different from the  $b$ -function for a polynomial in the sense of Kashiwara. For a homogeneous differential operator  $P(x, \partial) \in D(V)^G$ , any complex number can be a root of its  $b_P$ -function and the multiplicity can be also taken to be arbitrary. We shall prove it in the sequel.

**Proposition 3.2.** *Let  $P(x, \partial) \in D(V)^G$  be a homogeneous differential operator with homogeneous degree  $kn$  and  $b_P$ -function  $b_P(s)$ . Then we can construct a homogeneous differential operator with the same homogeneous degree  $kn$  the same  $b_P$ -function  $b_P(s)$  as a power product of the differential operators (24),  $\det(\partial^*)$  and  $\det(x)$ .*

*Proof.* Let  $\vartheta := \text{tr}(x \cdot \partial^*)$  be the Euler operator defined in (24). Then we have

$$\begin{aligned} \frac{1}{n}(\vartheta + n\lambda) \det(x)^s &= \frac{1}{n}(ns + n\lambda) \det(x)^s \\ &= (s + \lambda) \det(x)^s \end{aligned}$$

Then the polynomial

$$f(s) := \prod_{k=1}^l (s - \lambda_k)^{p_k}$$

with  $\lambda_1, \dots, \lambda_l \in \mathbb{C}$  and  $p_1, \dots, p_l \in \mathbb{Z}_{>0}$  is the  $b_P$ -function of the homogeneous differential operator

$$P(x, \partial) = \left(\frac{1}{n}\right)^p \prod_{k=1}^l (\vartheta + n\lambda_k)^{p_k}$$

of homogeneous degree 0 where  $p = p_1 + \dots + p_l$ . Indeed, we have

$$P(x, \partial) \det(x)^s = f(s) \det(x)^s.$$

If we need a homogeneous differential operator of positive homogeneous degree  $nq$  ( $q \in \mathbb{Z}_{>0}$ ) with  $b_P$ -function  $f(s)$ , we can take

$$P(x, \partial) = \det(x)^q \left(\frac{1}{n}\right)^p \prod_{k=1}^l (\vartheta + n\lambda_k)^{p_k}$$

and obtain

$$P(x, \partial) \det(x)^s = c \cdot f(s) \det(x)^{s+q}.$$

For a homogeneous differential operator of negative homogeneous degree  $-nq$  ( $q \in \mathbb{Z}_{>0}$ ), we have only to take

$$P(x, \partial) = \det(\partial^*)^q \left(\frac{1}{n}\right)^p \prod_{k=1}^l (\vartheta + n\lambda_k)^{p_k}.$$

Then we have

$$P(x, \partial) \det(x)^s = c \cdot f(s) b^q(s-1) \det(x)^{s-q}.$$

where  $b^{\mathbf{q}}(s-1) := b(s-1)b(s-2)\cdots b(s-q)$  with  $b(s) := \prod_{i=1}^n (s + \frac{i+1}{2})$ . The divisor  $b^{\mathbf{q}}(s-1)$  must be added to  $b_P$ -function because of Proposition 3.1-3.  $\square$

*Remark 3.1.* The explicit computation of  $b_P$ -functions for a given invariant differential operator  $P(x, \partial)$  is an important problem. The author [13] gives an algorithm to compute it explicitly. The method employed in [13] is to give a procedure to rewrite  $P(x, \partial)$  in terms of the invariant differential operators  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  defined in Example 2.1. Then, since we have computed the  $b_P$ -functions of  $P_i(x, \partial)$  ( $i = 1, \dots, n-1$ ),  $\det(x)$  and  $\det(\partial^*)$  in Example 3.1, we obtain the  $b_P$ -function of the given  $P(x, \partial)$ .

The algorithm in [13] is possible to be implemented on some computer algebra system. But the possibility of completion of the calculation fully depends on the performance of the computer.

#### 4. FIRST MAIN THEOREM AND ITS PROOF.

The purpose of this section is to prove the following theorem.

**Theorem 4.1.** *Let  $P(x, \partial) \in D(V)^G$  be a non-zero homogeneous differential operator with homogeneous degree  $kn$ . We suppose that*

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial). \quad (35)$$

*The space of  $G$ -invariant hyperfunction solutions of the differential equation  $P(x, \partial)u(x) = 0$  is finite dimensional. The solutions  $u(x)$  are given as finite linear combinations of quasi-homogeneous  $G$ -invariant hyperfunctions.*

*Proof.* Note that the functional equation

$$\mathfrak{M}_1 : \begin{cases} P(x, \partial)u(x) = 0, \\ u(x) \text{ is } \mathbf{SL}_n(\mathbb{R})\text{-invariant,} \end{cases} \quad (36)$$

and the system of linear differential equation

$$\mathfrak{M}_2 : \begin{cases} P(x, \partial)u(x) = 0, \\ \langle A \cdot x, \partial \rangle u(x) = 0 \text{ for all } A \in \mathfrak{sl}_n(\mathbb{R}), \end{cases} \quad (37)$$

are equivalent. Here,  $\mathfrak{sl}_n(\mathbb{R})$  is the Lie algebra of  $\mathbf{SL}_n(\mathbb{R})$ , the action of  $A \in \mathfrak{sl}_n(\mathbb{R})$  to  $x \in V = \text{Sym}_n(\mathbb{R})$  is  $A \cdot x := Ax + x^t A$  and  $\langle x, \xi \rangle := \text{tr}(x \cdot \xi)$  is a canonical bilinear form on  $(x, \xi) \in T^*V = V \times V^*$ , which is automatically extended to the complexification to  $(x, \xi) \in T^*V_{\mathbb{C}} = V_{\mathbb{C}} \times V_{\mathbb{C}}^*$ . We shall use  $\mathfrak{M}_2$  instead of  $\mathfrak{M}_1$  in the following.

**Lemma 4.2.** *Suppose the condition (35). Then the system of linear differential equation  $\mathfrak{M}_2$  is a holonomic system. Then the hyperfunction solution space of  $\mathfrak{M}_2$  is finite dimensional.*

*Proof.* In order to show that  $\mathfrak{M}_2$  is a holonomic system, we have only to prove that the characteristic variety of  $\mathfrak{M}_2$  is a complex Lagrangian subvariety

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in  $T^*V_{\mathbb{C}}$  where  $V_{\mathbb{C}}$  is a complexification of  $V$ . From the definition, the characteristic variety  $\text{ch}(\mathfrak{M}_2)$  of  $\mathfrak{M}_2$  is given by

$$\text{ch}(\mathfrak{M}_2) := \left\{ (x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \begin{array}{l} \sigma(P)(x, \xi) = 0 \text{ and } \langle A \cdot x, \xi \rangle = 0 \\ \text{for all } A \in \mathfrak{sl}_n(\mathbb{R}). \end{array} \right\} \quad (38)$$

since the differential operators in (37) form an involutive basis of the differential equation  $\mathfrak{M}_2$ . Let

$$W := \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \mathfrak{sl}_n(\mathbb{R})\}, \quad (39)$$

$$W_0 := \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \mathfrak{gl}_n(\mathbb{R})\}, \quad (40)$$

where  $\mathfrak{gl}_n(\mathbb{R})$  is the Lie algebra of  $\text{GL}_n(\mathbb{R})$ . From the definition, we have

$$W_0 = W \cap \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \langle x, \xi \rangle = 0\}. \quad (41)$$

Let  $T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}$  be the conormal bundle of  $S_{i\mathbb{C}} := \{x \in \text{Sym}_n(\mathbb{C}) \mid \text{rank}(x) = n - i\}$  and let  $\overline{T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}}$  be its Zariski-closure. Then, we have

$$W_0 = \bigcup_{i=0}^n \overline{T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}}, \quad (42)$$

and

$$\begin{aligned} W \cap \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \det(x) = 0\} &= \bigcup_{i=1}^n \overline{T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}} \subset W_0, \\ W \cap \{(x, \xi) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid \det(\xi) = 0\} &= \bigcup_{i=0}^{n-1} \overline{T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}} \subset W_0. \end{aligned} \quad (43)$$

Moreover, we can prove that

$$W - W_0 \text{ is a Zariski open dense subset in } W. \quad (44)$$

These results (42), (43) and (44) are obtained by computing the  $\text{GL}_n(\mathbb{C})$ -orbit structure of  $W$  explicitly (see the author's result [9, pp.400]). Since each  $\Lambda_{i\mathbb{C}} := \overline{T_{S_{i\mathbb{C}}}^* V_{\mathbb{C}}}$  is an irreducible Lagrangian subvariety in  $T^*V_{\mathbb{C}}$ ,  $W_0$  is a Lagrangian subvariety in  $T^*V_{\mathbb{C}}$ .

We prove Lemma 4.2 by showing that the characteristic variety  $\text{ch}(\mathfrak{M}_2)$  coincides with  $W_0$ . Before proving this, we need some arguments on the subvariety  $W$ ,  $W_0$  and  $W^\circ$ . Let

$$W^\circ := \{(x, s\partial^* \log \det(x)) \in V_{\mathbb{C}} \times V_{\mathbb{C}}^* \mid s \in \mathbb{C} - \{0\}, x \in V - S\}, \quad (45)$$

and let  $\overline{W^\circ}$  be its Zariski-closure. Here,  $\partial^*$  is a symmetric matrix of differential operator defined by (22). We shall prove that

$$W^\circ = W - W_0 \text{ and } \overline{W^\circ} = W, \quad (46)$$

It is clear that  $\overline{W^\circ} = W$  if  $W^\circ = W - W_0$  is valid since  $W - W_0$  is a Zariski open dense subset in  $W$ . So we have only to prove that  $W^\circ = W - W_0$ .

We first show that  $W^\circ \subset W - W_0$ . If  $(x_0, \xi_0) \in W^\circ$ , then  $\det(x_0) \neq 0$  and

$$\xi_0 = s_0 \partial^* \log \det(x)|_{x=x_0} = s_0 (x_0)^{-1}$$

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with some constant  $s_0 \in \mathbb{C}$ . Then for any  $A \in \mathfrak{sl}_n(\mathbb{R})$ , we have

$$\begin{aligned}
 \langle A \cdot x_0, \xi_0 \rangle &= \text{tr}(A \cdot x_0 \xi_0) = s_0 \text{tr}((A \cdot x_0)(x_0)^{-1}) \\
 &= s_0 \text{tr}((Ax_0 + x_0^t A)(x_0)^{-1}) \\
 &= s_0(\text{tr}(Ax_0(x_0)^{-1}) + \text{tr}((x_0^t A)(x_0)^{-1})) \\
 &= s_0(\text{tr}(A) + \text{tr}({}^t A)) = 0,
 \end{aligned} \tag{47}$$

and hence  $(x_0, \xi_0) \in W$ . On the other hand, since

$$\langle x_0, \xi_0 \rangle = \text{tr}(x_0 \xi_0) = s_0 \text{tr}(x_0(x_0)^{-1}) = \text{tr}(I_n) \neq 0,$$

we have  $(x_0, \xi_0) \notin W_0$ . Then  $W^\circ \subset W - W_0$  follows.

Next we prove that  $W^\circ \supset W - W_0$ . Suppose that  $(x_0, \xi_0) \in W - W_0$ . Then we have  $\det(x_0) \neq 0$ . In order to prove it, we assume that  $\det(x_0) = 0$ . Then there exists  $A \in \mathfrak{sl}_n(\mathbb{R})$  satisfying  $A \cdot x_0 = x_0$ . Therefore, we have

$$0 = \langle A \cdot x_0, \xi_0 \rangle = \langle x_0, \xi_0 \rangle,$$

since  $(x_0, \xi_0) \in W = \{(x, \xi) \mid \langle A \cdot x, \xi \rangle = 0 \text{ for all } A \in \mathfrak{sl}_n(\mathbb{R})\}$ . This means that  $(x_0, \xi_0) \in W_0$  and it violates the assumption that  $(x_0, \xi_0) \in W - W_0$ . Then  $\det(x_0) \neq 0$ .

Since  $\xi_0$  is not zero and contained in the orthogonal complement of the tangent subspace

$$\mathfrak{sl}_n(\mathbb{C}) \cdot x_0 = \left\{ \begin{array}{l} \text{the complex vector space gener-} \\ \text{ated by } A \cdot x_0 \text{ with } A \in \mathfrak{sl}_n(\mathbb{R}). \end{array} \right\} \subset TV_{\mathbb{C}},$$

it is a non-constant multiple of  $x_0^{-1}$ . In fact,  $x_0^{-1}$  is contained in the orthogonal complement of  $\mathfrak{sl}_n(\mathbb{C}) \cdot x_0$  by the same argument in (47). On the other hand, the dimension of  $\mathfrak{sl}_n(\mathbb{C}) \cdot x_0$  is  $n(n+1)/2 - 1$  since it is the tangent space at  $x_0$  of the subvariety  $\{x \in V_{\mathbb{C}} \mid \det(x) = \det(x_0)\}$ , which is an  $\text{SL}_n(\mathbb{C})$ -orbit of  $x_0$  in  $V_{\mathbb{C}}$ . Therefore, the orthogonal complement is one dimensional and it is generated by  $x_0^{-1}$  and hence  $\xi_0 = c(x_0)^{-1}$  with a non-zero constant  $c$ . Then we have

$$(x_0, \xi_0) = (x_0, c(x_0)^{-1}) \in W^\circ$$

if  $(x_0, \xi_0) \in W - W_0$ . This means  $W^\circ \supset W - W_0$ . Then, by combining the fact that  $W^\circ \subset W - W_0$  proved in the preceding paragraph, we have  $W^\circ = W - W_0$ .

We show that

$$s = \frac{1}{n} \langle x, \xi \rangle \Big|_{W^\circ} \tag{48}$$

on the subvariety  $W^\circ = W - W_0$ . Since

$$(x, \xi) = (x, s\partial^* \log \det(x)) = (x, sx^{-1})$$

on  $W^\circ = W - W_0$ , we have

$$\langle x, \xi \rangle = \langle x, sx^{-1} \rangle = \text{tr}(sxx^{-1}) = \text{tr}(sI_n) = sn,$$

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and hence we have (48). The function  $s = \frac{1}{n}\langle x, \xi \rangle|_{W^\circ}$  can be naturally extended to  $W = \overline{W - W_0} = \overline{W^\circ}$  and

$$W_0 = W \cap \{(x, \xi) \mid \langle x, \xi \rangle = 0\} = W \cap \{(x, \xi) \mid s = 0\}. \quad (49)$$

Now we go back to the proof of the fact that the characteristic variety  $\text{ch}(\mathfrak{M}_2)$  coincides with  $W_0$ . Let  $nk (k \in \mathbb{Z})$  be the homogeneous degree of  $P(x, \partial)$  and Let  $q (q \in \mathbb{Z}_{\geq 0})$  be the order of  $P(x, \partial)$ . We denote by  $\sigma(P)(x, \xi)$  the principal symbol of  $P(x, \partial)$ . By restricting  $P(x, \partial)$  to  $W^\circ$ , we have

$$\sigma(P)(x, s\partial^* \log \det(x)) = \sigma(P)(x, sx^{-1}) = s^q \sigma(P)(x, x^{-1}).$$

On the other hand, we have

$$\begin{aligned} & P(x, \partial) \det(x)^s \\ &= s^q \sigma(P)(x, \partial^* \det(x)) \det(x)^{s-q} + (\text{lower degree terms in } s) \\ &= s^q \sigma(P)(x, \det(x)^{-1} \partial^* \det(x)) \det(x)^s + (\text{lower degree terms in } s) \\ &= s^q \det(x)^{-k} \sigma(P)(x, x^{-1}) \det(x)^{s+k} + (\text{lower degree terms in } s) \\ &= b_P(s) \det(x)^{s+k} \end{aligned}$$

From the assumption (35), the  $b_P$ -function is given by

$$b_P(s) = b_0 s^q + b_1 s^{q-1} + \cdots + b_q$$

with  $b_0 \neq 0$ . Then we have  $\det(x)^{-k} \sigma(P)(x, x^{-1}) = b_0 \neq 0$  and hence

$$\sigma(P)(x, x^{-1}) = b_0 \det(x)^k.$$

Then by considering  $\sigma(P)(x, \xi)$  on  $W^\circ$ , we have  $(x, \xi) = (x, sx^{-1})$  and

$$\sigma(P)(x, \xi)|_{W^\circ} = s^q \sigma(P)(x, x^{-1})|_{W^\circ} = s^q b_0 \det(x)^k|_{W^\circ}.$$

If  $k \geq 0$ , then  $\sigma(P)(x, \xi)$  is extended to  $W$  naturally as  $s^q b_0 \det(x)^k$ . Then

$$\begin{aligned} \text{ch}(\mathfrak{M}_2) &= W \cap \{(x, \xi) \mid \sigma(P)(x, \xi) = 0\} = W \cap \{(x, \xi) \mid s^q b_0 \det(x)^k = 0\} \\ &= (W \cap \{(x, \xi) \mid s = 0\}) \cup (W \cap \{(x, \xi) \mid \det(x) = 0\}), \end{aligned}$$

and, by (49) and (43), we have  $\text{ch}(\mathfrak{M}_2) = W_0$ . If  $k \leq 0$ , then  $q \geq -nk$  and

$$\begin{aligned} \sigma(P)(x, \xi)|_{W^\circ} &= s^q \sigma(P)(x, x^{-1})|_{W^\circ} = s^q b_0 \det(x)^k|_{W^\circ} \\ &= s^q b_0 \det(s\xi^{-1})^k|_{W^\circ} = s^{q+nk} b_0 \det(\xi)^{-k}|_{W^\circ} \end{aligned}$$

since  $(x, \xi) = (x, sx^{-1})$  on  $W^\circ$ . Then  $\sigma(P)(x, \xi)$  is extended to  $W$  naturally as  $s^{q+nk} b_0 \det(\xi)^{-k}$  and

$$\begin{aligned} \text{ch}(\mathfrak{M}_2) &= W \cap \{(x, \xi) \mid \sigma(P)(x, \xi) = 0\} = W \cap \{(x, \xi) \mid s^{q+nk} b_0 \det(\xi)^{-k} = 0\} \\ &= (W \cap \{(x, \xi) \mid s = 0\}) \cup (W \cap \{(x, \xi) \mid \det(\xi) = 0\}), \end{aligned}$$

and, by (49) and (43), we have  $\text{ch}(\mathfrak{M}_2) = W_0$ .

Thus we complete the proof.  $\square$

**Lemma 4.3.** *Let  $Sol(\mathfrak{M}_2)$  be the hyperfunction solution space to the system of linear differential equation  $\mathfrak{M}_2$ . Then the Euler operator  $\vartheta := \text{tr}(x\partial^*)$  is a linear endomorphism on the finite dimensional complex vector space  $Sol(\mathfrak{M}_2)$ .*

*Proof.* This is clear since  $\vartheta$  is commutative with the differential operators  $P(x, \partial)$  and  $\langle A \cdot x, \partial \rangle (A \in \mathfrak{sl}_n(\mathbb{R}))$ .  $\square$

Now we go back to the proof of Theorem 4.1. Let  $f$  be the dimension of the vector space  $\mathfrak{M}_2$  and consider the linear map

$$\vartheta : Sol(\mathfrak{M}_2) \longrightarrow Sol(\mathfrak{M}_2).$$

We can choose a basis  $\{u_i(x)\}_{i=1,\dots,f}$  of  $Sol(\mathfrak{M}_2)$  so that the matrix expression of the linear map  $\vartheta$  with respect to  $\{u_i(x)\}_{i=1,\dots,f}$  is a Jordan's canonical form. Then, for each  $u_i(x)$ , there exist an eigenvalue  $\lambda_i$  and a non-negative integer  $k_i$  satisfying

$$\vartheta \begin{bmatrix} u_i(x) \\ u_{i+1}(x) \\ \vdots \\ \vdots \\ u_{i+k_i-1}(x) \\ u_{i+k_i}(x) \end{bmatrix} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 1 & 0 \\ \vdots & \cdots & \cdots & 0 & \lambda_i & 1 \\ 0 & \cdots & \cdots & 0 & 0 & \lambda_i \end{bmatrix} \begin{bmatrix} u_i(x) \\ u_{i+1}(x) \\ \vdots \\ \vdots \\ u_{i+k_i-1}(x) \\ u_{i+k_i}(x) \end{bmatrix}$$

From this equation, we have

$$(\vartheta - \lambda_i)^{k_i+1} u_i(x) = 0,$$

which means that  $u_i(x)$  is a  $G$ -invariant quasi-homogeneous hyperfunction. This is what we have to prove (see Definition 1.2).  $\square$

## 5. SOME PROPERTIES OF LAURENT EXPANSION COEFFICIENTS OF COMPLEX POWERS OF DETERMINANT FUNCTION.

The following theorem is well-known, see, for example, [11]. The hyperfunction  $P^{[\vec{a}, s]}(x)$  with a meromorphic parameter  $s \in \mathbb{C}$  has the following functional equation (50).

**Proposition 5.1.** *Let  $\partial^*$  be the symmetric matrix of differential operators defined by (22).*

1. *We have*

$$\begin{aligned} (\det(\partial^*)) P^{[\vec{a}, s+1]}(x) &= b(s) \cdot P^{[\vec{a}^\#, s]}(x) \\ &= b(s) \cdot (\det(x)) \cdot P^{[\vec{a}, s-1]}(x) \end{aligned} \tag{50}$$

with  $\vec{a}^\# = \vec{a}^{\#1} := ((-1)^n a_0, \dots, -a_{n-1}, a_n)$  and

$$b(s) = c \cdot (s+1) \left(s + \frac{3}{2}\right) \cdots \left(s + \frac{n+1}{2}\right), \tag{51}$$

where  $c$  is a constant.

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2.  $P^{[\vec{a},s]}(x)$  is holomorphic with respect to  $s \in \mathbb{C}$  except for the poles at  $s = -(k+1)/2$  with  $k = 1, 2, \dots$ . The possible highest order of the pole of  $P^{[\vec{a},s]}(x)$  at  $s = -(k+1)/2$  is

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & (k = 1, 2, \dots, n-1), \\ \lfloor \frac{n}{2} \rfloor & (k = n, n+1, \dots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & (k = n, n+1, \dots, \text{ and } k+n \text{ is even}). \end{cases} \quad (52)$$

- Proof.* 1. This is a special case of Proposition 3.1-2, and the  $b_P$ -function for  $\det(\partial^*)$  in (51) is well known.  
2. This is also well known (See also [12]). □

Here we give two definitions.

**Definition 5.1** (possible highest order). Let  $\lambda \in \mathbb{C}$  be a fixed complex number.

1. We denote by  $PHO(\lambda)$  the possible highest order of the pole of  $P^{[\vec{a},s]}(x)$  at  $s = \lambda$ . Namely we define

$$PHO(\lambda) := \begin{cases} \lfloor \frac{k+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = 1, 2, \dots, n-1), \\ \lfloor \frac{n}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = n, n+1, \dots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & \lambda = -\frac{k+1}{2} \ (k = n, n+1, \dots, \text{ and } k+n \text{ is even}), \\ 0 & \text{otherwise.} \end{cases} \quad (53)$$

2. Let  $q \in \mathbb{Z}$ . We define a vector subspace  $A(\lambda, q)$  of  $\mathbb{C}^{n+1}$  by

$$A(\lambda, q) := \{ \vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a},s]}(x) \text{ has a pole of order } \leq q \text{ at } s = \lambda \}. \quad (54)$$

Then we have  $A(\lambda, q-1) \subset A(\lambda, q)$  by definition. We define  $\overline{A(\lambda, q)}$  by

$$\overline{A(\lambda, q)} := A(\lambda, q) / A(\lambda, q-1) \quad (55)$$

It is easily verified that  $\overline{A(\lambda, q)} = \{0\}$  if  $q > PHO(\lambda)$  or  $q < 0$ . We have

$$\bigoplus_{q \in \mathbb{Z}} \overline{A(\lambda, q)} = \bigoplus_{0 \leq q \leq PHO(\lambda)} \overline{A(\lambda, q)} \simeq \mathbb{C}^{n+1}. \quad (56)$$

In particular,  $\vec{a} = 0$  if  $\vec{a} \in A(\lambda, q)$  for some  $q < 0$  since  $A(\lambda, q) = \{0\}$  for  $q < 0$ . However, when  $q < 0$ , a pole of order  $q$  means a zero of order  $-q$ .

**Definition 5.2** (Laurent expansion coefficients). Let  $\lambda \in \mathbb{C}$  be a fixed complex number.

1. We define  $o(\vec{a}, \lambda) \in \mathbb{Z}$  by

$$o(\vec{a}, \lambda) := \text{the order of pole of } P^{[\vec{a},s]}(x) \text{ at } s = \lambda. \quad (57)$$



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Then  $o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$ . We have  $p = o(\vec{a}, \lambda)$  if and only if  $\vec{a} \in A(\lambda, p)$  and  $[\vec{a}] \in \overline{A(\lambda, p)}$  is not zero.

2. Let  $\vec{a} \in \mathbb{C}^{n+1}$  and let  $r = o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$ . This means that  $P^{[\vec{a}, s]}(x)$  has a pole of order  $r$  at  $s = \lambda$ . Then we have the Laurent expansion of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$ ,

$$P^{[\vec{a}, s]}(x) = \sum_{w=-r}^{\infty} P_w^{[\vec{a}, \lambda]}(x) (s - \lambda)^w. \quad (58)$$

We often denote by

$$\text{Laurent}_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x)) := P_w^{[\vec{a}, \lambda]}(x) \quad (59)$$

the  $w$ -th Laurent expansion coefficient of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$  in (58). It is easily checked that  $P_w^{[\vec{a}, \lambda]}(x)$  is linear with respect to  $\vec{a} \in \mathbb{C}^{n+1}$ .

We shall investigate some properties of  $P^{[\vec{a}, s]}(x)$  and their Laurent expansion coefficients  $P_w^{[\vec{a}, \lambda]}(x)$  at  $s = \lambda$ . First we show the following lemma.

**Lemma 5.2.** *For  $\vec{a} \in \mathbb{C}^{n+1}$ , let  $r = o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$  be the order of pole of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$  and let*

$$P^{[\vec{a}, s]}(x) = \sum_{w \in \mathbb{Z}_{\geq 0}} P_w^{[\vec{a}, \lambda]}(x) (s - \lambda)^w$$

*be the Laurent expansion of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$ . Then we have*

$$\frac{1}{n}(\vartheta - n\lambda)P_{w+1}^{[\vec{a}, \lambda]}(x) = P_w^{[\vec{a}, \lambda]}(x) \quad (60)$$

*for all  $w \in \mathbb{Z}$  and hence  $P_w^{[\vec{a}, \lambda]}(x) \neq 0$  for all  $w \geq -r$  and  $P_w^{[\vec{a}, \lambda]}(x) = 0$  for all  $w < -r$ . In addition, we have  $(\vartheta - n\lambda)^{i+1}P_{-r+i}^{[\vec{a}, \lambda]}(x) = 0$  and  $(\vartheta - n\lambda)^i P_{-r+i}^{[\vec{a}, \lambda]}(x) \neq 0$  for  $i = 1, 2, \dots$ , where  $\vartheta := \text{tr}(x\partial^*)$ .*

*Proof.* Note that

$$\frac{1}{n}(\vartheta - n\lambda)P^{[\vec{a}, s]}(x) = (s - \lambda)P^{[\vec{a}, s]}(x).$$

Then we have

$$\sum_{w \in \mathbb{Z}} \frac{1}{n}(\vartheta - n\lambda)P_w^{[\vec{a}, \lambda]}(x)(s - \lambda)^w = \sum_{w \in \mathbb{Z}} P_w^{[\vec{a}, \lambda]}(x)(s - \lambda)^{w+1},$$

and hence

$$\frac{1}{n}(\vartheta - n\lambda)P_{w+1}^{[\vec{a}, \lambda]}(x) = P_w^{[\vec{a}, \lambda]}(x)$$

for all  $w \in \mathbb{Z}$ . Therefore, if  $P_{w+1}^{[\vec{a}, \lambda]}(x) = 0$ , then  $P_w^{[\vec{a}, \lambda]}(x) = 0$ , and if  $P_w^{[\vec{a}, \lambda]}(x) \neq 0$ , then  $P_{w+1}^{[\vec{a}, \lambda]}(x) \neq 0$ . Since  $P_{-r-1}^{[\vec{a}, \lambda]}(x) = 0$  and  $P_{-r}^{[\vec{a}, \lambda]}(x) \neq 0$  from the assumption, we have the results by applying (60) repeatedly.  $\square$

Then we have the following proposition.

**Proposition 5.3.** *Let  $\vec{a}, \vec{b} \in \mathbb{C}^{n+1}$  and let  $r = PHO(\lambda)$ .*

1. *Let  $q$  be an integer in  $q \leq r$ . We have*

$$\vec{a} - \vec{b} \in A(\lambda, q)$$

*if and only if*

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a},s]}(x)) = Laurent_{s=\lambda}^{(w)}(P^{[\vec{b},s]}(x))$$

*for  $w = -r, -r + 1, \dots, -q - 1$ . In particular,*

$$P^{[\vec{a},s]}(x) = P^{[\vec{b},s]}(x)$$

*if  $\vec{a} - \vec{b} \in A(\lambda, q)$  for some  $q < 0$ .*

2. *Let  $\vec{a}_1, \dots, \vec{a}_k \in \mathbb{C}^{n+1}$  be the vectors satisfying that they are linearly independent in the quotient space  $\mathbb{C}^{n+1}/A(\lambda, q-1)$  with a positive integer  $q$ . Then, for an integer  $w$  with  $w \geq -q$ , the hyperfunctions*

$$\{Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}_i,s]}(x))\}_{i=1,2,\dots,k}$$

*are linearly independent.*

*Proof.* 1. If  $\vec{a} - \vec{b} \in A(\lambda, q)$ , then  $P^{[\vec{a}-\vec{b},s]}(x)$ 's order of pole at  $s = \lambda$  is less than  $q$ . By expanding the both sides of

$$P^{[\vec{a}-\vec{b},s]}(x) = P^{[\vec{a},s]}(x) - P^{[\vec{b},s]}(x)$$

as Laurent expansions, we have

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}-\vec{b},s]}(x)) = 0 \quad \text{if } w < -q,$$

and hence

$$Laurent_{s=\lambda}^{(w)}(P^{[\vec{a},s]}(x)) = Laurent_{s=\lambda}^{(w)}(P^{[\vec{b},s]}(x))$$

for  $w < -q$ . In particular, if  $q < 0$  and  $\vec{a} - \vec{b} \in A(\lambda, q)$ , then  $P^{[\vec{a}-\vec{b},s]}(x)$  has a zero at  $s = \lambda$ , which means  $\vec{a} - \vec{b} = 0$ . Then we have  $\vec{a} = \vec{b}$ .

2. For an integer  $w \geq -q$ , if

$$\sum_{i=1}^k c_i P_w^{[\vec{a}_i,\lambda]}(x) = \sum_{i=1}^k P_w^{[c_i \vec{a}_i,\lambda]}(x) = 0,$$

then  $\sum_{i=1}^k P^{[c_i \vec{a}_i,s]}(x)$ 's order of pole at  $s = \lambda$  is strictly less than  $q$  by Lemma 5.2. Then  $\sum_{i=1}^k c_i \vec{a}_i \in A(\lambda, q-1)$  and hence  $\sum_{i=1}^k c_i \vec{a}_i$  is zero in the quotient space  $\mathbb{C}^{n+1}/A(\lambda, q-1)$ . Since  $\vec{a}_i (i = 1, \dots, k)$  are linearly independent in  $\mathbb{C}^{n+1}/A(\lambda, q-1)$ , we have  $c_1 = \dots = c_k = 0$ . Then

$$\{Laurent_{s=\lambda}^{(w)}(P^{[\vec{a}_i,s]}(x)) = P_w^{[\vec{a}_i,\lambda]}(x)\}_{i=1,2,\dots,k}$$

are linearly independent.

For each  $\lambda \in \mathbb{C}$ , if  $P^{[\vec{a},s]}(x)$  does not have a pole at  $s = \lambda$ , then  $P^{[\vec{a},\lambda]}(x) = P^{[\vec{a},s]}(x)|_{s=\lambda}$  is well-defined and a non-zero homogeneous hyperfunction of homogeneous degree  $\lambda n$ . If  $P^{[\vec{a},s]}(x)$  has a pole at  $s = \lambda$  of order  $p$ , then  $(s - \lambda)^p P^{[\vec{a},s]}(x)|_{s=\lambda}$  is a non-zero homogeneous hyperfunction of homogeneous degree  $\lambda n$ . Furthermore, as we have remarked in Remark 1.2, we can prove that Laurent expansion coefficients of  $P^{[\vec{a},s]}(x)$  are quasi-homogeneous hyperfunctions and the converse is also true. We shall prove it in the following Theorem 5.6. Before proving the theorem, we show the following Lemma 5.4. This is a consequence of the author's paper [11].

We define a standard basis of  $\mathbb{C}^{n+1}$ .

**Definition 5.3** (Standard basis). Let

$$SB := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\} \quad (61)$$

be a basis of  $\mathbb{C}^{n+1}$ . We say that  $SB$  is a *standard basis* of  $\mathbb{C}^{n+1}$  at  $s = \lambda$  if the following property holds: there exists an increasing integer sequence

$$0 < k(0) < k(1) < \dots < k(Pho(\lambda)) = n \quad (62)$$

such that

$$SB_q := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_{k(q)}\}$$

is a basis of  $A(\lambda, q)$  for each  $q$  in  $0 \leq q \leq Pho(\lambda)$ . It is easily seen that the representatives of  $SB_q - SB_{q-1}$  form a basis of the quotient vector space  $\overline{A(\lambda, q)} := A(\lambda, q)/A(\lambda, q-1)$ .

We need the following lemma which is essentially proved in [11].

**Lemma 5.4.** *Let  $v(x)$  be a  $G$ -invariant homogeneous hyperfunction of degree  $n\lambda$ , i.e., quasi-homogeneous of degree  $n\lambda$  and of quasi-degree 0 and let  $\{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\}$  be a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda$ . Then  $v(x)$  can be expressed uniquely as*

$$v(x) = \sum_{i=0}^n c_i \text{Laurent}_{s=\lambda}^{(-o(\vec{a}_i, \lambda))} (P^{[\vec{a}_i, s]}(x))$$

with suitable  $c_i \in \mathbb{C}$  ( $i = 0, \dots, n$ ) where  $o(\vec{a}_i, \lambda)$  is the order of pole of  $P^{[\vec{a}_i, s]}(x)$  at  $s = \lambda$ . In other words, the elements

$$\{\text{Laurent}_{s=\lambda}^{(-o(\vec{a}_i, \lambda))} (P^{[\vec{a}_i, s]}(x))\}_{i=0, \dots, n}$$

are linearly independent and form a basis of the space of hyperfunctions that are  $G$ -invariant and homogeneous of degree  $n\lambda$ .

*Proof.* In the author's paper [11, Theorem 5.6], he proved that

1. The dimension of  $G$ -invariant homogeneous hyperfunctions of homogeneous degree  $n\lambda$  is  $n + 1$ .

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2. Any  $G$ -invariant homogeneous hyperfunction of homogeneous degree  $n\lambda$  is written as

$$\sum_{i=0}^n c_i(s) |P(x)|_i^s \Big|_{s=\lambda}, \quad (63)$$

where  $c_i(s)$  are meromorphic functions defined at  $s = \lambda$ .

Then we can write as

$$v(x) = \sum_{i=0}^n c_i(s) |P(x)|_i^s \Big|_{s=\lambda},$$

with  $c_i(s) = \sum_{j \in \mathbb{Z}} c_{ij}(s - \lambda)^j$  are meromorphic functions near  $s = \lambda$ .

We see that  $c_i(s)$ 's are assumed to be holomorphic near  $s = \lambda$ . Indeed, the Laurent expansion of  $|P(x)|_i^s$  is given by

$$\sum_{w \in \mathbb{Z}} P_w^{[\vec{e}_i, \lambda]}(x) (s - \lambda)^w$$

where  $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$  is the unit vector only whose  $i$ -th entry is 1. Then we have

$$\begin{aligned} \sum_{i=0}^n c_i(s) |P(x)|_i^s &= \sum_{i=0}^n \sum_{j \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} c_{ij} P_w^{[\vec{e}_i, \lambda]}(x) (s - \lambda)^{j+w} \\ &= \sum_{i=0}^n \sum_{k \in \mathbb{Z}} \sum_{k=j+w} c_{ij} P_w^{[\vec{e}_i, \lambda]}(x) (s - \lambda)^k \\ &= \sum_{i=0}^n \sum_{k \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} c_{i, k-w} P_w^{[\vec{e}_i, \lambda]}(x) (s - \lambda)^k \\ &= \sum_{k \in \mathbb{Z}} \sum_{w \in \mathbb{Z}} P_w^{[\sum_{i=0}^n c_{i, k-w} \vec{e}_i, \lambda]}(x) (s - \lambda)^k \end{aligned}$$

By putting  $\vec{b}_{k-w} := \sum_{i=0}^n c_{i, k-w} \vec{e}_i$  Hence we have

$$\sum_{w \in \mathbb{Z}} P_w^{[\vec{b}_{k-w}, \lambda]}(x) = 0,$$

for all  $k < 0$  and

$$v(x) = \sum_{w \in \mathbb{Z}} P_w^{[\vec{b}_{-w}, \lambda]}(x). \quad (64)$$

If  $w < 0$ , then  $\text{Supp}(P_w^{[\vec{a}, \lambda]}(x)) \subset S$  (see Remark 2.1), and hence we have,

$$\begin{aligned} \sum_{w \in \mathbb{Z}} P_w^{[\vec{b}_{k-w}, \lambda]}(x)|_{V-S} &= \sum_{\substack{w \in \mathbb{Z} \\ w \geq 0}} P_w^{[\sum_{i=0}^n c_{i,k-w} \vec{e}_i, \lambda]}(x)|_{V-S} \\ &= \sum_{\substack{w \in \mathbb{Z} \\ w \geq 0}} \sum_{i=0}^n c_{i,k-w} P_w^{[\vec{e}_i, \lambda]}(x)|_{V-S} \\ &= \sum_{\substack{w \in \mathbb{Z} \\ w \geq 0}} \sum_{i=0}^n c_{i,k-w} |P(x)|_i^\lambda (\log |P(x)|)^w|_{V-S} = 0 \end{aligned}$$

for any  $k < 0$ . Since the hyperfunctions in

$$\left\{ |P(x)|_i^\lambda (\log |P(x)|)^w \right\}_{\substack{i=0, \dots, n \\ w=0, 1, \dots}}$$

are linearly independent, we have  $c_{i,k-w} = 0$  for all  $i = 0, \dots, n$ ,  $k = -1, -2, \dots$  and  $w = 0, 1, \dots$ . This means that

$$c_{i,j} = 0 \text{ for all } i = 0, \dots, n \text{ and } j = -1, -2, \dots$$

Therefore, we may assume that each  $c_i(s)$  is holomorphic at  $s = \lambda$  and  $\vec{b}_j = 0$  for  $j = -1, -2, \dots$

By (64), we have

$$v(x) = \sum_{\substack{w < 0 \\ w \in \mathbb{Z}}} P_w^{[\vec{b}_{-w}, \lambda]}(x) = \sum_{-PHO(\lambda) \leq w \leq 0} P_w^{[\vec{b}_{-w}, \lambda]}(x).$$

We shall show that each  $P_w^{[\vec{b}_{-w}, \lambda]}(x)$  is homogeneous of degree  $n\lambda$ . Indeed, since  $v(x)$  is homogeneous of degree  $n\lambda$  by definition, we have

$$\frac{1}{n}(\vartheta - n\lambda) \sum_{-PHO(\lambda) \leq w \leq 0} P_w^{[\vec{b}_{-w}, \lambda]}(x) = \sum_{-PHO(\lambda) \leq w \leq 0} P_{w-1}^{[\vec{b}_{-w}, \lambda]}(x) = 0.$$

by (60). The non-zero hyperfunctions in

$$\{P_{w-1}^{[\vec{b}_{-w}, \lambda]}(x) \mid -PHO(\lambda) \leq w \leq 0\}$$

are linearly independent since their support are dimensionally different, i.e.,

$$\dim(\text{Supp}(P_{w_1}^{[\vec{a}_1, \lambda]}(x))) < \dim(\text{Supp}(P_{w_2}^{[\vec{a}_2, \lambda]}(x)))$$

are if  $\text{Supp}(P_{w_1}^{[\vec{a}_1, \lambda]}(x)) \neq 0$ ,  $\text{Supp}(P_{w_2}^{[\vec{a}_2, \lambda]}(x)) \neq 0$  and  $-PHO(\lambda) \leq w_1 < w_2 \leq 0$ , by Theorem A.2 in Appendix A. Then we have

$$\frac{1}{n}(\vartheta - n\lambda) P_w^{[\vec{b}_{-w}, \lambda]}(x) = P_{w-1}^{[\vec{b}_{-w}, \lambda]}(x) = 0$$

for each  $-PHO(\lambda) \leq w \leq 0$ . Therefore, if  $P_w^{[\vec{b}_{-w}, \lambda]}(x) \neq 0$ , then  $o(\vec{b}_{-w}, \lambda) =$

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Using the standard basis  $SB := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\}$  defined by (61),  $SB_q = \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_{k(q)}\}$  is a basis of  $A(\lambda, q)$  and  $SB_q - SB_{q-1} = \{\vec{a}_{k(q-1)+1}, \dots, \vec{a}_{k(q)}\}$  is a basis of  $\overline{A(\lambda, q)}$ . In the sum

$$v(x) = \sum_{-PHO(\lambda) \leq w \leq 0} P_w^{[\vec{b}_{-w}, \lambda]}(x),$$

if  $P_w^{[\vec{b}_{-w}, \lambda]}(x) \neq 0$ , then  $o(\vec{b}_{-w}, \lambda) = -w$  and  $\vec{b}_{-w} \in A(\lambda, -w)$ , and hence we can write

$$\vec{b}_{-w} = \sum_{i=k(-w-1)+1}^{k(-w)} c_i \vec{a}_i + (\text{a linear sum of } \vec{a}_i \text{ in } i = 0, \dots, k(-w-1)).$$

Since  $\vec{a}_i \in A(\lambda, -w-1)$  for  $i = 0, \dots, k(-w-1)$  and  $w = -o(\vec{a}_i, \lambda)$  for  $i = k(-w-1)+1, \dots, k(-w)$ , we have

$$P_w^{[\vec{b}_{-w}, \lambda]}(x) = P_w^{[\sum_{i=k(-w-1)+1}^{k(-w)} c_i \vec{a}_i, \lambda]}(x) = \sum_{i=k(-w-1)+1}^{k(-w)} c_i P_{-o(\vec{a}_i, \lambda)}^{[\vec{a}_i, \lambda]}(x)$$

Then we have

$$\begin{aligned} v(x) &= \sum_{\substack{-PHO(\lambda) \leq w \leq 0 \\ PHO(\lambda)}} P_w^{[\vec{b}_{-w}, \lambda]}(x) \\ &= \sum_{-w=0}^{PHO(\lambda)} \sum_{i=k(-w-1)+1}^{k(-w)} c_i P_{-o(\vec{a}_i, \lambda)}^{[\vec{a}_i, \lambda]}(x) \\ &= \sum_{i=0}^n c_i P_{-o(\vec{a}_i, \lambda)}^{[\vec{a}_i, \lambda]}(x) \end{aligned}$$

by defining  $k(-1) = -1$  and  $c_i = 0$  for  $i = k(-w-1)+1, \dots, k(-w)$  if  $P_w^{[\vec{b}_{-w}, \lambda]}(x) = 0$ . This is what we want to prove.  $\square$

By using standard basis of  $\mathbb{C}^{n+1}$ , we have the following proposition.

**Proposition 5.5.** *Let  $SB := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\}$  be a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda$  and let  $r_j := o(\vec{a}_j, \lambda) \in \mathbb{Z}_{\geq 0}$ . Then the Laurent expansion coefficients at  $s = \lambda$*

$$\{Laurent_{s=\lambda}^{(-r_j+i)}(P^{[\vec{a}_j, s]}(x))\}_{i=0,1,2,\dots \text{ and } j=0,1,2,\dots,n} \quad (65)$$

are linearly independent.

*Proof.* We have only to show that the elements of the finite subset

$$\{Laurent_{s=\lambda}^{(-r_j+i)}(P^{[\vec{a}_j, s]}(x))\}_{i=0,1,2,\dots,k \text{ and } j=0,1,2,\dots,n} \quad (66)$$

of (65) are linearly independent. We shall prove it by induction on the number  $k$ . If  $k = 0$ , we see that the elements of (66) are linearly independent

by Lemma 5.4. Next we suppose that it is true when  $k \geq 0$  and that

$$\sum_{i=0}^{k+1} \sum_{j=0}^n c_{ij} \text{Laurent}_{s=\lambda}^{(-r_j+i)}(P^{[\vec{a}_j, s]}(x)) = 0 \quad (67)$$

where  $c_{ij}$  are constants. Then we have

$$\begin{aligned} & \left(\frac{1}{n}(\vartheta - n\lambda)\right) \sum_{i=0}^{k+1} \sum_{j=0}^n c_{ij} \text{Laurent}_{s=\lambda}^{(-r_j+i)}(P^{[\vec{a}_j, s]}(x)) \\ &= \sum_{i=1}^{k+1} \sum_{j=0}^n c_{ij} \text{Laurent}_{s=\lambda}^{(-r_j+i-1)}(P^{[\vec{a}_j, s]}(x)) \\ &= \sum_{i=0}^k \sum_{j=0}^n c_{i+1, j} \text{Laurent}_{s=\lambda}^{(-r_j+i)}(P^{[\vec{a}_j, s]}(x)) = 0 \end{aligned}$$

by Lemma 5.2. Then, by the induction hypothesis, we have

$$c_{i+1, j} = 0 \quad \text{for all } i = 0, \dots, k \text{ and } j = 0, \dots, n.$$

Then, by (67), we have

$$\sum_{j=0}^n c_{0, j} \text{Laurent}_{s=\lambda}^{(-r_j)}(P^{[\vec{a}_j, s]}(x)) = 0, \quad (68)$$

and hence, by Lemma 5.4, we have

$$c_{0, j} = 0 \quad \text{for all } j = 0, \dots, n.$$

Thus we complete the proof by induction.  $\square$

**Theorem 5.6.** *Let  $r := o(\vec{a}, \lambda) \in \mathbb{Z}_{\geq 0}$  be the order of the pole of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$ .*

1. *Then the Laurent expansion coefficient of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$  defined by (59)*

$$\text{Laurent}_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x)) = P_w^{[\vec{a}, \lambda]}(x)$$

*is a quasi-homogeneous hyperfunction of degree  $n\lambda$  of quasi-degree  $r + w$ . Conversely, let  $v(x) \in QH(n\lambda)^G$ , the space of  $G$ -invariant quasi-homogeneous hyperfunctions (Definition 1.2). Then  $v(x)$  is written as a linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda$ .*

2. *Let*

$$LC(\lambda, w) := \left\{ \begin{array}{l} \text{the vector space generated by} \\ \{ \text{Laurent}_{s=\lambda}^{(w)}(P^{[\vec{a}, s]}(x)) \mid \vec{a} \in \mathbb{C}^{n+1} \} \end{array} \right\}, \quad (69)$$

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i.e., the vector space of  $w$ -th Laurent expansion coefficients of  $P^{[\vec{a},s]}(x)$  at  $s = \lambda$ . Then we have the direct sum decomposition

$$QH(n\lambda)^G = \bigoplus_{\substack{w \in \mathbb{Z}, \\ w \geq -PHO(\lambda)}} LC(\lambda, w) \quad (70)$$

*Proof.* 1. It is clear that  $P_w^{[\vec{a},\lambda]}(x)$  is a quasi-homogeneous hyperfunction because we have

$$(\vartheta - n\lambda)^{r+w+1} P_w^{[\vec{a},\lambda]}(x) = 0$$

by Lemma 5.2.

We prove the converse by induction on the quasi-degree of  $v(x) \in QH(n\lambda)^G$ . First we suppose that  $v(x)$ 's quasi-degree is 0, i.e.,  $v(x)$  is homogeneous of degree  $n\lambda$ . Then, by Lemma 5.4,  $v(x)$  is written as a linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda$ .

Next we suppose that  $v(x)$  is written as a linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda$  if  $v(x) \in QH(n\lambda)^G$  is of quasi-degree is  $q - 1$ . We shall prove this is true even if  $v(x)$  is of quasi-degree is  $q$ . Let

$$v_0(x) := \left(\frac{1}{n}(\vartheta - n\lambda)\right)^q v(x).$$

Then, by Definition 1.2, we have  $(\frac{1}{n}(\vartheta - n\lambda))v_0(x) = 0$ , and hence, by Lemma 5.4,  $v_0(x)$  is written as

$$v_0(x) = \sum_{i=0}^n c_i \text{Laurent}_{s=\lambda}^{(-o(\vec{a}_i, \lambda))} (P^{[\vec{a}_i, s]}(x)) = \sum_{i=0}^n c_i P_{-o(\vec{a}_i, \lambda)}^{[\vec{a}_i, \lambda]}(x)$$

by using a standard basis  $\{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\}$  of  $\mathbb{C}^{n+1}$  at  $s = \lambda$  and constants  $c_i \in \mathbb{C}$ . By putting

$$v_1(x) := v(x) - \sum_{i=0}^n c_i P_{-o(\vec{a}_i, \lambda)+q}^{[\vec{a}_i, \lambda]}(x),$$

we have

$$\left(\frac{1}{n}(\vartheta - n\lambda)\right)^q v_1(x) = v_0(x) - \sum_{i=0}^n c_i P_{-o(\vec{a}_i, \lambda)}^{[\vec{a}_i, \lambda]}(x) = 0$$

by applying (60)  $q$  times. Then  $v_1(x) \in QH(n\lambda)^G$  and it is of quasi-degree is  $q - 1$ . By the induction hypothesis,  $v_1(x)$  is written as a linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda$ , and so is

$$v(x) = v_1(x) + \sum_{i=0}^n c_i P_{-o(\vec{a}_i, \lambda)+q}^{[\vec{a}_i, \lambda]}(x).$$

Thus we complete the proof by induction on the quasi-degree.



2. We have seen that the vector spaces

$$LC(\lambda, w) \quad (w \in \mathbb{Z} \text{ and } w \geq -PHO(\lambda))$$

are linearly independent by Proposition 5.5 since  $LC(\lambda, w)$  is generated by non-zero Laurent expansion coefficients  $P_w^{[\vec{a}_i, \lambda]}(x)$  where  $\{\vec{a}_0, \dots, \vec{a}_n\}$  is a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda$ . Then we have the result.  $\square$

By combining Theorem 4.1 and Theorem 5.6, we have the following corollary.

**Corollary 5.7.** *Let  $P(x, \partial) \in D(V)^G$  be a non-zero homogeneous differential operator with homogeneous degree  $kn$  satisfying the condition (35). Then  $G$ -invariant hyperfunction solutions  $u(x)$  to the differential equation  $P(x, \partial)u(x) = 0$  are written as finite linear combinations of Laurent expansion coefficients of  $|P(x)|_i^s$  at a finite number of points.*

## 6. SECOND MAIN THEOREMS AND THEIR PROOFS.

The purpose of this section is to prove the following theorems.

**Theorem 6.1.** *Let  $P(x, \partial) \in D(V)^G$  be a non-zero homogeneous differential operator with homogeneous degree  $kn$  and let  $v(x)$  be a quasi-homogeneous  $G$ -invariant hyperfunction of homogeneous degree  $n\lambda$ . We suppose that*

$$b_P(s) \not\equiv 0. \quad (71)$$

*Then*

1. *We can construct a  $G$ -invariant hyperfunction solution  $u(x) \in \mathcal{B}(V)^G$  to the differential equation  $P(x, \partial)u(x) = v(x)$ , which is given as a sum of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda - k$  and hence is quasi-homogeneous of degree  $n(\lambda - k)$ .*
2. *Any  $G$ -invariant hyperfunction solution  $u(x)$  is given as finite linear combinations of quasi-homogeneous  $G$ -invariant hyperfunctions, and hence it is written as a finite linear combinations of Laurent expansion coefficients of  $|P(x)|_i^s$  at a finite number of points in  $\mathbb{C}$ .*

*Proof.* The second statement is derived from the first statement by Theorem 4.1 and Lemma 5.4. Indeed, we see that any  $G$ -invariant hyperfunction solution to the differential equation  $P(x, \partial)u(x) = v(x)$  is a sum of several quasi-homogeneous  $G$ -invariant hyperfunctions, and hence it is written as a finite linear combinations of Laurent expansion coefficients of  $|P(x)|_i^s$  at a finite number of points.

We shall prove the first statement. Let  $P(x, \partial)$  be a  $G$ -invariant homogeneous differential operator of homogeneous degree  $nk$ . For a  $G$ -invariant quasi-homogeneous hyperfunction  $v(x)$  of homogeneous degree  $n\lambda$ , we have

$$v(x) \in QH(n\lambda)^G = \bigoplus_{\substack{w \in \mathbb{Z} \\ w \geq PHO(\lambda)}} LC(\lambda, w).$$

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By Theorem 5.6,  $v(x)$  is written as a finite sum of the hyperfunctions which are given as Laurent expansion coefficients of  $P^{[\vec{a},s]}(x)$  at  $s = \lambda$ :  $P_w^{[\vec{a},\lambda]}(x)$  with some  $w \in \mathbb{Z}$  and some  $\vec{a} \in \mathbb{C}^{n+1}$ . Thus we have only to show Theorem 6.1 when  $v(x) = P_w^{[\vec{a},\lambda]}(x)$ .

By (27), we have

$$P(x, \partial)P^{[\vec{a},s]}(x) = b_P(s)P^{[\vec{a}^{\#k}, s+k]}(x) \quad (72)$$

where  $b_P(s)$  is the  $b_P$ -function of  $P(x, \partial)$ . By expanding the both sides of (72) to Laurent series at  $s = \lambda$ , we have

$$\begin{aligned} P(x, \partial)P^{[\vec{a},s]}(x) &= P(x, \partial) \sum_{w \in \mathbb{Z}} P_w^{[\vec{a},\lambda]}(x)(s - \lambda)^w \\ &= b_P(s)P^{[\vec{a}^{\#k}, s+k]}(x) \\ &= b_P(s) \sum_{w' \in \mathbb{Z}} P_{w'}^{[\vec{a}^{\#k}, \lambda+k]}(x)(s - \lambda)^{w'}. \end{aligned}$$

Since  $b_P(s) \neq 0$ , we can divide it as

$$b_P(s) = (s - \lambda)^p \tilde{b}(s) \text{ with } \tilde{b}(\lambda) \neq 0.$$

Then  $\tilde{b}(s)^{-1}$  is holomorphic at  $s = \lambda$  and expanded to Taylor series

$$\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i(s - \lambda)^i.$$

We have

$$\begin{aligned} &P(x, \partial) \sum_{w \in \mathbb{Z}} \left( \sum_{i+j=w} b_i P_j^{[\vec{a},\lambda]}(x) \right) (s - \lambda)^w \\ &= P(x, \partial) \sum_{j \in \mathbb{Z}} \tilde{b}(s)^{-1} P_j^{[\vec{a},\lambda]}(x) (s - \lambda)^j \\ &= (s - \lambda)^p \sum_{w' \in \mathbb{Z}} P_{w'}^{[\vec{a}^{\#k}, \lambda+k]}(x) (s - \lambda)^{w'} \\ &= \sum_{w \in \mathbb{Z}} P_{w-p}^{[\vec{a}^{\#k}, \lambda+k]}(x) (s - \lambda)^w \end{aligned} \quad (73)$$

Comparing the both sides of (73), we obtain

$$P(x, \partial) \left( \sum_{i+j=w} b_i P_j^{[\vec{a},\lambda]}(x) \right) = P_{w-p}^{[\vec{a}^{\#k}, \lambda+k]}(x)$$

for each  $w \in \mathbb{Z}$ . By arranging the indices we have

$$P(x, \partial) \left( \sum_{i+j=w+p} b_i P_j^{[\vec{a}^{\#k}, \lambda+k]}(x) \right) = P_w^{[\vec{a}, \lambda']}(x),$$

with  $\lambda' = \lambda + k$ , which is what we have to prove. Then we can construct a  $G$ -invariant hyperfunction solution  $u(x)$  to

$$P(x, \partial)u(x) = P_w^{[\vec{a}, \lambda']}(x),$$

which is written as finite linear combinations of Laurent expansion coefficients of  $|P(x)|_i^s$  at  $s = \lambda' - k = \lambda$ . This is true for any  $G$ -invariant solution to

$$P(x, \partial)u(x) = v(x)$$

where  $v(x)$  is a  $G$ -invariant quasi-homogeneous hyperfunction.  $\square$

Next we consider the construction of  $G$ -invariant hyperfunction solutions to  $P(x, \partial)u(x) = 0$ . By Theorem 4.1, a  $G$ -invariant hyperfunction solution  $u(x)$  is written as

$$u(x) = u_1(x) + \cdots + u_p(x) \quad (74)$$

where each  $u_i(x)$  is a quasi-homogeneous hyperfunctions of homogeneous degree  $n\lambda_i$  and  $\lambda_i \in \mathbb{C}$  are mutually different complex numbers. Then we have

$$P(x, \partial)u_i(x) = 0 \quad \text{for each } i = 1, 2, \dots, p. \quad (75)$$

Indeed, we see that

$$P(x, \partial)u(x) = P(x, \partial)u_1(x) + \cdots + P(x, \partial)u_p(x) = 0$$

where the homogeneous degree of each  $P(x, \partial)u_i(x)$  is  $n\lambda_i + nk$ . If some of  $P(x, \partial)u_i(x)$  ( $i = 1, \dots, p$ ) are not zero, then they are zero since they are linearly independent. This is a contradiction. Then we have (75). Then we have only to construct quasi-homogeneous  $G$ -invariant hyperfunction solution of homogeneous degree  $n\lambda$ , which is written as a finite linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$

**Theorem 6.2.** *Let  $P(x, \partial) \in D(V)^G$  be a non-zero homogeneous differential operator of homogeneous degree  $kn$  satisfying the condition (35). Then we can construct the  $G$ -invariant quasi-homogeneous hyperfunction solution of homogeneous degree  $n\lambda$  to the differential equation  $P(x, \partial)u(x) = 0$  as a finite linear combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$ . It is determined by the homogeneous degree  $kn$  and  $b_P(s)$  and does not depend on  $P(x, \partial)$  itself.*

*Proof.* Let  $P(x, \partial)$  be a non-zero homogeneous differential operator of homogeneous degree  $kn$  and whose  $b_P$ -function is  $b_P(s)$ . Then we have

$$P(x, \partial)P^{[\vec{a}, s]}(x) = b_P(s)P^{[\vec{a}^{\#k}, s+k]}(x).$$

We expand the both sides into the Laurent series. By the Laurent expansions

$$\begin{aligned} b_P(s) &= \sum_{i \in \mathbb{Z}} b_i (s - \lambda)^i, \\ P^{[\vec{a}, s]}(x) &= \sum_{w \in \mathbb{Z}} P_w^{[\vec{a}, \lambda]}(x) (s - \lambda)^w, \\ P^{[\vec{a}^{\#k}, s+k]}(x) &= \sum_{j \in \mathbb{Z}} P_j^{[\vec{a}^{\#k}, \lambda+k]}(x) (s - \lambda)^j, \end{aligned}$$

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we have

$$P(x, \partial) \sum_{w \in \mathbb{Z}} P_w^{[\vec{a}, \lambda]}(x) (s - \lambda)^w = \sum_{w \in \mathbb{Z}} \sum_{i+j=w} b_i P_j^{[\vec{a}^{\#k}, \lambda+k]}(x) (s - \lambda)^w.$$

Then we have

$$P(x, \partial) P_w^{[\vec{a}, \lambda]}(x) = \sum_{i+j=w} b_i P_j^{[\vec{a}^{\#k}, \lambda+k]}(x).$$

When  $u(x)$  is given as a quasi-homogeneous hyperfunction of degree  $n\lambda$ , it is written as a finite sum

$$u(x) = \sum_{p=1}^q P_{w_p}^{[\vec{a}_p, \lambda]}(x) \quad (76)$$

with  $w_p \in \mathbb{Z}$  and  $\vec{a}_p \in \mathbb{C}^{n+1}$ . Then

$$\begin{aligned} P(x, \partial)u(x) &= \sum_{p=1}^q \left( \sum_{i+j=w_p} b_i P_j^{[\vec{a}_p^{\#k}, \lambda+k]}(x) \right) = \sum_{p=1}^q \sum_{j \in \mathbb{Z}} b_{w_p-j} P_j^{[\vec{a}_p^{\#k}, \lambda+k]}(x) \\ &= \sum_{j \in \mathbb{Z}} \sum_{p=1}^q b_{w_p-j} P_j^{[\vec{a}_p^{\#k}, \lambda+k]}(x) = \sum_{j \in \mathbb{Z}} P_j^{[\sum_{p=1}^q b_{w_p-j} \vec{a}_p^{\#k}, \lambda+k]}(x) \\ &= \sum_{j \in \mathbb{Z}} P_j^{[\vec{c}_j, \lambda+k]}(x), \end{aligned}$$

where  $\vec{c}_j := \sum_{p=1}^q b_{w_p-j} \vec{a}_p^{\#k}$ . This is a finite sum since  $\vec{c}_j = 0$  for sufficiently large  $|j|$ . By Theorem 5.6, non-zero elements in  $\{P_j^{[\vec{c}_j, \lambda+k]}(x) \mid j \in \mathbb{Z}\}$  are linearly independent. Then  $P(x, \partial)u(x) = 0$  is equivalent to that

$$P_j^{[\vec{c}_j, \lambda+k]}(x) = 0 \text{ for all } j \in \mathbb{Z}. \quad (77)$$

Thus we can construct a solution  $u(x)$  as a function of the form (76) satisfying the condition (77). The condition (77) depends only on  $k$  and  $b_P(s)$ . Then the condition for  $\mathbf{G}$ -invariant  $u(x)$  to be annihilated by  $P(x, \partial)$  depends only on  $k$  and  $b_P(s)$ . □

**Corollary 6.3.** *Let  $P(x, \partial), Q(x, \partial) \in D(V)^{\mathbf{G}}$  be non-zero homogeneous differential operators with the same homogeneous degree and satisfying the condition (35). We suppose that their  $b_P$ -functions coincide with each other. Then the  $\mathbf{G}$ -invariant hyperfunction solution space of the differential equation  $P(x, \partial)u(x) = v(x)$  coincides with that of  $Q(x, \partial)u(x) = v(x)$ .*

*Proof.* We have seen that in the proof of Theorem 6.1 that the differential equations  $P(x, \partial)u(x) = v(x)$  and  $Q(x, \partial)u(x) = v(x)$  have the same  $\mathbf{G}$ -invariant solution if their homogeneous degrees and  $b_P$ -functions coincide with each other. On the other hand, by Theorem 6.2, the  $\mathbf{G}$ -invariant solution spaces of  $P(x, \partial)u(x) = 0$  and  $Q(x, \partial)u(x) = 0$  coincide under the same conditions. Since any solution to  $P(x, \partial)u(x) = v(x)$  (resp.  $Q(x, \partial)u(x) =$

$v(x)$ ) are given by a sum of one solution to  $P(x, \partial)u(x) = v(x)$  (resp.  $Q(x, \partial)u(x) = v(x)$ ) and one solution to  $P(x, \partial)u(x) = 0$  (resp.  $Q(x, \partial)u(x) = 0$ ), the  $\mathbf{G}$ -invariant hyperfunction solution space of the differential equation  $P(x, \partial)u(x) = v(x)$  coincides with that of  $Q(x, \partial)u(x) = v(x)$ .  $\square$

## 7. ORDERS OF POLES OF COMPLEX POWERS OF DETERMINANT FUNCTIONS.

In the preceding section, we have proved that the solutions of (2) can be constructed in terms of the Laurent expansion coefficients of the complex powers of the determinant functions  $P^{[\vec{a}, s]}(x)$  (Theorem 6.1 and Theorem 6.2). However, in order to apply these constructions of solutions in concrete examples, we have to see the exact pole of  $P^{[\vec{a}, s]}(x)$  especially at  $s = \text{half-integers } \frac{1}{2}\mathbb{Z}$ .

In this section, we shall give a condition to determine the exact order of pole of  $P^{[\vec{a}, s]}(x)$  at  $s = \text{half-integers } \frac{1}{2}\mathbb{Z}$  for a given vector  $\vec{a} \in \mathbb{C}^{n+1}$ . This is a direct application of the author's result in [12].

In order to determine the exact pole of  $P^{[\vec{a}, s]}(x)$  at  $s = s_0$ , the author introduced the coefficient vectors

$$\mathbf{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1} \quad (78)$$

with  $k = 0, 1, \dots, n$  in [12]. Here,  $(\mathbb{C}^{n+1})^*$  means the dual vector space of  $\mathbb{C}^{n+1}$ . Each element of  $\mathbf{d}^{(k)}[s_0]$  is a linear form on  $\vec{a} \in \mathbb{C}^{n+1}$  depending on  $s_0 \in \mathbb{C}$ , i.e., a linear map from  $\mathbb{C}^{n+1}$  to  $\mathbb{C}$ ,

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \mathbb{C}. \quad (79)$$

We denote

$$\langle \mathbf{d}^{(k)}[s_0], \vec{a} \rangle = (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \dots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \mathbb{C}^{n-k+1}. \quad (80)$$

The precise definition of  $\mathbf{d}^{(k)}[s_0]$  is the following.

**Definition 7.1** (Coefficient vectors  $\mathbf{d}^{(k)}[s_0]$ ). Let  $s_0$  be a *half-integer*, i.e., a rational number given by  $q/2$  with an integer  $q$ . We define the *coefficient vectors*  $\mathbf{d}^{(k)}[s_0]$  ( $k = 0, 1, \dots, n$ ) by induction in the following way.

1. First, we set

$$\mathbf{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \dots, d_n^{(0)}[s_0]) \quad (81)$$

such that  $\langle d_i^{(0)}[s_0], \vec{a} \rangle := a_i$  for  $i = 0, 1, \dots, n$ .

2. Next, we define  $\mathbf{d}^{(1)}[s_0]$  and  $\mathbf{d}^{(2)}[s_0]$  by

$$\mathbf{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \dots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^n, \quad (82)$$

with  $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0]d_{j+1}^{(0)}[s_0]$ , and

$$\mathbf{d}^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \dots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1}, \quad (83)$$

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with  $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$ . Here,

$$\epsilon[s_0] := \begin{cases} 1 & , \text{ (if } s_0 \text{ is a strict half-integer),} \\ (-1)^{s_0+1} & , \text{ (if } s_0 \text{ is an integer).} \end{cases} \quad (84)$$

A *strict half-integer* means a rational number given by  $q/2$  with an odd integer  $q$ .

3. Lastly, by induction on  $k$ , we define the coefficient vectors  $\mathbf{d}^{(k)}[s_0]$  for  $k = 0, 1, \dots, n$  by

$$\mathbf{d}^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \dots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l}, \quad (85)$$

with  $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$ , and

$$\mathbf{d}^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \dots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1}, \quad (86)$$

with  $d_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$ .

By using  $\mathbf{d}^{(k)}[s_0]$  in Definition 7.1, the author obtained an algorithm to compute the exact order of poles of  $P^{[\vec{a}, s]}(x)$  in [12]. It is Theorem A.1 in Appendix. In this section, we shall characterize

$$A(\lambda, q) := \{\vec{a} \in \mathbb{C}^{n+1} \mid P^{[\vec{a}, s]}(x) \text{ has a pole of order } \leq q \text{ at } s = \lambda\}. \quad (87)$$

in terms of the coefficient vectors  $\mathbf{d}^{(k)}[\lambda]$ .

**Definition 7.2.** We define the vector subspaces  $D_{half}^{(l)}$ ,  $D_{even}^{(l)}$  and  $D_{odd}^{(l)}$  in  $\mathbb{C}^{n+1}$ .

1. Note that  $\mathbf{d}^{(2l+2)}[\lambda]$  does not depend on the choice of  $\lambda$  if it is a half-integer. We define

$$D_{half}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \mathbf{d}^{(2l+2)}[\lambda], \vec{a} \rangle = 0 \text{ for any strict half-integer } \lambda\}.$$

2. Note that  $\mathbf{d}^{(2l+1)}[\lambda]$  does not depend on the choice of  $\lambda$  if it is an odd integer or an even integer, respectively. We define

$$D_{odd}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \mathbf{d}^{(2l+1)}[\lambda], \vec{a} \rangle = 0 \text{ for any odd integer } \lambda\}.$$

$$D_{even}^{(l)} := \{\vec{a} \in \mathbb{C}^{n+1} \mid \langle \mathbf{d}^{(2l+1)}[\lambda], \vec{a} \rangle = 0 \text{ for any even integer } \lambda\}.$$

**Theorem 7.1.**  $D_{half}^{(l)}$ ,  $D_{even}^{(l)}$  and  $D_{odd}^{(l)}$  in  $\mathbb{C}^{n+1}$  have the following properties.

1. We define

$$\vec{a}^\# = \vec{a}^{\#1} := ((-1)^n a_0, (-1)^{n-1} a_1, \dots, a_n) \in \mathbb{C}^{n+1}$$

for  $\vec{a} = (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$ . Then we have

$$\vec{a} \in D_{odd}^{(l)} \iff \vec{a}^\# \in D_{even}^{(l)}$$

and

$$\vec{a} \in D_{half}^{(l)} \iff \vec{a}^\# \in D_{half}^{(l)}.$$

2. Let  $l$  be an integer  $0 \leq l < PHO(\lambda)$ . Then we have

$$\vec{a} \in A(\lambda, l) \iff \begin{cases} \vec{a} \in D_{half}^{(l)} & \text{if } \lambda \text{ is a strict half-integer,} \\ \vec{a} \in D_{odd}^{(l)} & \text{if } \lambda \text{ is an odd integer,} \\ \vec{a} \in D_{even}^{(l)} & \text{if } \lambda \text{ is an even integer.} \end{cases} \quad (88)$$

In addition, we have  $A(\lambda, PHO(\lambda)) = \mathbb{C}^{n+1}$ .

*Proof.* We can see that the second statement is nothing but the definition of  $D_{half}^{(l)}$ ,  $D_{even}^{(l)}$  and  $D_{odd}^{(l)}$  by Theorem A.1 in Appendix, which is the main result of [12].

We shall prove the first statement. Let **odd** be an odd integer and let **even** be an even integer. We have only to prove that

$$\langle \mathbf{d}^{(2l+1)}[\mathbf{odd}], \vec{a} \rangle = (-1)^n \langle \mathbf{d}^{(2l+1)}[\mathbf{even}], \vec{a}^\# \rangle. \quad (89)$$

for each  $l = 0, 1, 2, \dots$ . We prove it by induction on  $l$ . When  $l = 0$ , we have

$$\begin{aligned} \langle \mathbf{d}^{(1)}[\mathbf{odd}], \vec{a} \rangle &= (a_0 + a_1, a_1 + a_2, \dots, a_{n-1} + a_n) \\ &= (-1)^n (a_0^\# - a_1^\#, a_1^\# - a_2^\#, \dots, a_{n-1}^\# - a_n^\#) \\ &= (-1)^n \langle \mathbf{d}^{(1)}[\mathbf{even}], \vec{a}^\# \rangle \end{aligned}$$

since  $\vec{a}^\# = (a_0^\#, a_1^\#, \dots, a_n^\#) = ((-1)^n a_0, (-1)^{n-1} a_1, \dots, a_n)$ . We see that

$$\langle \mathbf{d}^{(2l+1)}[\mathbf{odd}], \vec{a} \rangle = (-1)^n \langle \mathbf{d}^{(2l+1)}[\mathbf{even}], \vec{a}^\# \rangle$$

if

$$\langle \mathbf{d}^{(2l-1)}[\mathbf{odd}], \vec{a} \rangle = (-1)^n \langle \mathbf{d}^{(2l-1)}[\mathbf{even}], \vec{a}^\# \rangle$$

by the definition of (85). Thus (89) is valid for all  $l = 0, 1, 2, \dots$  by induction on  $l$ . By (89), we have

$$\begin{aligned} \vec{a} \in D_{odd}^{(l)} &\iff \langle \mathbf{d}^{(2l+1)}[\mathbf{odd}], \vec{a} \rangle = 0 \\ &\iff \langle \mathbf{d}^{(2l+1)}[\mathbf{even}], \vec{a}^\# \rangle = 0 \iff \vec{a}^\# \in D_{even}^{(l)} \end{aligned}$$

Next let **half** be a strict half-integer. We have only to prove that

$$\langle \mathbf{d}^{(2l+2)}[\mathbf{half}], \vec{a} \rangle = \langle \mathbf{d}^{(2l+2)}[\mathbf{half}], \vec{a}^\# \rangle^\#. \quad (90)$$

for each  $l = 0, 1, 2, \dots$ . We prove it by induction on  $l$ . When  $l = 0$ , we have

$$\begin{aligned} \langle \mathbf{d}^{(2)}[\mathbf{half}], \vec{a} \rangle &= (a_0 + a_2, a_1 + a_3, \dots, a_{n-2} + a_n) \\ &= ((-1)^{n-2} (a_0^\# + a_2^\#), (-1)^{n-3} (a_1^\# + a_3^\#), \dots, (a_{n-2}^\# + a_n^\#)) \\ &= \langle \mathbf{d}^{(1)}[\mathbf{half}], \vec{a}^\# \rangle^\# \end{aligned}$$

since  $\vec{a}^\# = (a_0^\#, a_1^\#, \dots, a_n^\#) = ((-1)^n a_0, (-1)^{n-1} a_1, \dots, a_n)$ . We see that

$$\langle \mathbf{d}^{(2l+2)}[\mathbf{half}], \vec{a} \rangle = \langle \mathbf{d}^{(2l+2)}[\mathbf{half}], \vec{a}^\# \rangle^\#$$

if

$$\langle d^{(2l)}[half], \vec{a} \rangle = \langle d^{(2l)}[half], \vec{a}^\# \rangle^\#$$

by the definition of (86). Thus (90) is valid for all  $l = 0, 1, 2, \dots$  by induction on  $l$ . By (90), we have

$$\begin{aligned} \vec{a} \in D_{half}^{(l)} &\iff \langle d^{(2l+2)}[half], \vec{a} \rangle = 0 \\ &\iff \langle d^{(2l+2)}[half], \vec{a}^\# \rangle^\# = 0 \\ &\iff \langle d^{(2l+2)}[half], \vec{a}^\# \rangle = 0 \iff \vec{a}^\# \in D_{half}^{(l)} \end{aligned}$$

□

When  $\lambda \notin \frac{1}{2}\mathbb{Z}$ , any basis is a standard basis defined by Definition 5.3 since all  $P^{[\vec{a}, s]}(x)$  is holomorphic at  $s = \lambda$ . When  $\lambda$  is in  $\frac{1}{2}\mathbb{Z}$ , we can easily choose one standard basis for a given  $\lambda$  by utilizing Theorem 7.1. However, it is sufficient only to consider the following three kinds of standard basis,  $SB^{half}$ ,  $SB^{even}$  and  $SB^{odd}$ .

**Definition 7.3.** For  $\lambda \in \frac{1}{2}\mathbb{Z}$ , we define the bases of  $\mathbb{C}^{n+1}$   $SB^{half}$ ,  $SB^{even}$  and  $SB^{odd}$  by

$$\begin{aligned} SB^{half} &:= \{\vec{a}_0^{half}, \vec{a}_1^{half}, \dots, \vec{a}_n^{half}\} \text{ if } \lambda \text{ is a strict half-integer,} \\ SB^{even} &:= \{\vec{a}_0^{even}, \vec{a}_1^{even}, \dots, \vec{a}_n^{even}\} \text{ if } \lambda \text{ is an even integer,} \\ SB^{odd} &:= \{\vec{a}_0^{odd}, \vec{a}_1^{odd}, \dots, \vec{a}_n^{odd}\} \text{ if } \lambda \text{ is an odd integer,} \end{aligned} \quad (91)$$

satisfying that there exists an increasing integer sequence

$$0 < l(0) < l(1) < \dots < l(p) = n \quad (92)$$

with

$$p := \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } i+n \text{ is odd,} \\ \lfloor \frac{n+1}{2} \rfloor & \text{if } i+n \text{ is even,} \end{cases}$$

such that

$$\begin{aligned} SB_q^{half} &:= \{\vec{a}_0^{half}, \vec{a}_1^{half}, \dots, \vec{a}_{l(q)}^{half}\} \text{ is a basis of } D_{half}^{(q)}, \\ SB_q^{even} &:= \{\vec{a}_0^{even}, \vec{a}_1^{even}, \dots, \vec{a}_{l(q)}^{even}\} \text{ is a basis of } D_{even}^{(q)}, \\ SB_q^{odd} &:= \{\vec{a}_0^{odd}, \vec{a}_1^{odd}, \dots, \vec{a}_{l(q)}^{odd}\} \text{ is a basis of } D_{odd}^{(q)}, \end{aligned}$$

for  $q = 0, 1, \dots, p$ , respectively. In particular, we take  $SB^{even}$  and  $SB^{odd}$  such that

$$\vec{a}_j^{odd} = \vec{a}_j^{even\#} \quad (j = 0, 1, \dots, n) \quad (93)$$

where  $\vec{a}^\# := ((-1)^n a_0, (-1)^{n-1} a_1, \dots, a_n)$  for  $\vec{a} := (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$ . This is possible by Theorem 7.1.

**Proposition 7.2.** The bases (91) are standard bases for  $\lambda \in \frac{1}{2}\mathbb{Z}$  in the sense of Definition 5.3. When  $\lambda \notin \frac{1}{2}\mathbb{Z}$ , every basis is a standard basis since every  $P^{[\vec{a}, s]}(x)$  does not have a pole.



*Proof.* This is just the definition of the standard basis.  $\square$

## 8. ALGORITHMS FOR CONSTRUCTING SOLUTIONS — KERNELS OF $P(x, \partial)$ .

In this section we give algorithms to compute all the hyperfunction solutions to  $P(x, \partial)u(x) = 0$  for a homogeneous  $G$ -invariant differential operator  $P(x, \partial)$ .

**Algorithm 8.1** (The case of homogeneous degree zero). *For a given non-zero  $\mathrm{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)_0^G$  of homogeneous degree 0 satisfying the condition*

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (94)$$

*one algorithm to compute a basis of the  $\mathrm{SL}_n(\mathbb{R})$ -invariant differential equation  $P(x, \partial)u(x) = 0$  is given in the following.*

**Input:** A non-zero  $\mathrm{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)_0^G$  satisfying the condition (94).

**Output:** A basis of the  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation  $P(x, \partial)u(x) = 0$ .

**Procedure:**

1. Compute the  $b_P$ -function for  $P(x, \partial)$ . It is denoted by

$$b_P(s) = (s - \lambda_1)^{p_1} \cdots (s - \lambda_l)^{p_l}.$$

2. For each  $\lambda_i$  ( $i = 1, \dots, l$ ), take one standard basis at  $s = \lambda_i$

$$SB^{\lambda_i} = \{\tilde{a}_0(\lambda_i), \dots, \tilde{a}_n(\lambda_i)\},$$

which is defined in Definition 5.3.

3. Compute the Laurent expansion coefficients

$$\text{Laurent}_{s=\lambda_i}^{(k)}(P[\tilde{a}_j(\lambda_i), s](x))$$

for each  $\tilde{a}_j(\lambda_i)$  ( $i = 1, \dots, l, j = 0, \dots, n$ ) and  $k \in \mathbb{Z}$  in  $-o_{ij} \leq k \leq -o_{ij} + p_i - 1$  with  $o_{ij} := o(\tilde{a}_j(\lambda_i), \lambda_i)$ . Here,  $o(\tilde{a}, \lambda)$  has been defined by (57). Then we have the generators of the vector space  $L_{ij}$  in (95).

$L_{ij} :=$  the vector space generated by

$$\{\text{Laurent}_{s=\lambda_i}^{(k)}(P[\tilde{a}_j(\lambda_i), s](x))\}_{k=-o_{ij}, \dots, -o_{ij}+p_i-1} \quad (95)$$

4. Then

$$\bigoplus_{\substack{i=1, \dots, l \\ j=0, \dots, n}} L_{ij} \quad (96)$$

forms a basis of the  $G$ -invariant hyperfunction solution space to  $P(x, \partial)u(x) = 0$ .

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*Proof.* Note that, by Theorem 4.1 and Corollary 5.7, every  $G$ -invariant hyperfunction solution to  $P(x, \partial)u(x) = 0$  is written as a finite combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ). Suppose that  $u(x)$  is written as

$$u(x) = u_1(x) + \dots + u_l(x)$$

where each  $u_i(x)$  is quasi-homogeneous of degree  $s_i$  and  $s_1, \dots, s_l$  are mutually different. If  $P(x, \partial)u(x) = 0$ , then  $P(x, \partial)u_i(x) = 0$  for all  $i = 1, \dots, l$  since the homogeneous degrees of  $P(x, \partial)u_i(x)$  ( $i = 1, \dots, l$ ) are mutually different and hence linearly independent. Then, for each complex number  $\lambda \in \mathbb{C}$ , we have only to see what  $u(x)$  given as a finite combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$  is annihilated by  $P(x, \partial)$ .

Let

$$SB := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_n\}$$

be a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda$  with an increasing sequence

$$0 < k(0) < k(1) < \dots < k(Pho(\lambda)) = n \quad (97)$$

such that

$$SB_q := \{\vec{a}_0, \vec{a}_1, \dots, \vec{a}_{k(q)}\}$$

is a basis of  $A(\lambda, q)$  for each  $q$  in  $0 \leq q \leq Pho(\lambda)$ . Then every  $u(x)$  given as a finite combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$  is expressed as a linear combination

$$u(x) = \sum_{\substack{f, g \in \mathbb{Z} \\ 0 \leq g \leq n}} c_{f, g} P_f^{[\vec{a}_g, \lambda]}(x) \quad (98)$$

with

$$P_f^{[\vec{a}_g, \lambda]}(x) = \text{Laurent}_{s=\lambda}^{(f)}(P^{[\vec{a}_g, s]}(x))$$

and  $c_{f, g} \in \mathbb{C}$ . Let

$$b_P(s) = \sum_{i=0}^q b_i (s - \lambda)^{p+i}$$

be a expansion of  $b_P(s)$  with respect to  $(s - \lambda)$ . The number  $p$  is the multiplicity of  $b_P(s)$  at  $s = \lambda$ . Then what we have to prove is that

$$\begin{aligned} c_{f, g} &= 0 \text{ except for } -o(\vec{a}_g, \lambda) \leq f \leq -o(\vec{a}_g, \lambda) + p - 1 \\ &\text{if and only if } P(x, \partial)u(x) = 0 \end{aligned} \quad (99)$$

since  $P_f^{[\vec{a}_g, \lambda]}(x) = 0$  if  $f < -o(\vec{a}_g, \lambda)$  from the definition. Here,  $o(\vec{a}, \lambda)$  has been defined by (57). Indeed, the basis of  $L_{ij}$  in (95) is just the basis of the remainder terms in the expression (98) with the condition (99) when  $\lambda = \lambda_i$  and  $p = k_i$ . In particular, if  $\lambda$  is not a root of  $b_P(s) = 0$ , i.e.,  $p = 0$ , then

there is no  $G$ -invariant solution to  $P(x, \partial)u(x) = 0$ . The rest of the proof is devoted to proving (99).

The Laurent expansion of  $P^{[\vec{a}, s]}(x)$  at  $s = \lambda$  is denoted by

$$P^{[\vec{a}, s]}(x) = \sum_{w \in \mathbb{Z}} P_w^{[\vec{a}, \lambda]}(x)(s - \lambda)^w.$$

Then we have

$$\begin{aligned} P(x, \partial)P^{[\vec{a}, s]}(x) &= \sum_{w \in \mathbb{Z}} P(x, \partial)P_w^{[\vec{a}, \lambda]}(x)(s - \lambda)^w = b_P(s)P^{[\vec{a}, s]}(x) \\ &= \left( \sum_{i=0}^q b_i(s - \lambda)^{p+i} \right) \left( \sum_{j \in \mathbb{Z}} P_j^{[\vec{a}, \lambda]}(x)(s - \lambda)^j \right) \\ &= \sum_{w \in \mathbb{Z}} \sum_{i+j+p=w} b_i P_j^{[\vec{a}, \lambda]}(x)(s - \lambda)^w \end{aligned}$$

and hence we have

$$P(x, \partial)P_w^{[\vec{a}, \lambda]}(x) = \sum_{i+j+p=w} b_i P_j^{[\vec{a}, \lambda]}(x). \quad (100)$$

Here  $b_i = 0$  except for  $i$  in  $0 \leq i \leq q$  and  $P_j^{[\vec{a}, \lambda]}(x) = 0$  for sufficiently small  $j$ . Then for

$$u(x) = \sum_{\substack{f, g \in \mathbb{Z} \\ 0 \leq g \leq n}} c_{f, g} P_f^{[\vec{a}_g, \lambda]}(x),$$

we have

$$\begin{aligned} P(x, \partial)u(x) &= \sum_{f, g \in \mathbb{Z}} c_{f, g} P(x, \partial)P_f^{[\vec{a}_g, \lambda]}(x) = \sum_{f, g \in \mathbb{Z}} c_{f, g} \sum_{i=0}^q b_i P_{f-p-i}^{[\vec{a}_g, \lambda]}(x) \\ &= \sum_{f, g \in \mathbb{Z}} c_{f, g} \sum_{j \in \mathbb{Z}} b_{f-p-j} P_j^{[\vec{a}_g, \lambda]}(x) = \sum_{j \in \mathbb{Z}} \sum_{f, g \in \mathbb{Z}} c_{f, g} b_{f-p-j} P_j^{[\vec{a}_g, \lambda]}(x) \\ &= \sum_{j \in \mathbb{Z}} P_j^{[\sum_{f, g \in \mathbb{Z}} c_{f, g} b_{f-p-j} \vec{a}_g, \lambda]}(x) = 0 \end{aligned}$$

where  $g$  runs in  $0 \leq g \leq n$ . Then we have

$$\sum_{f, g \in \mathbb{Z}} c_{f, g} b_{f-p-j} \vec{a}_g = \sum_{g=0}^n \sum_{f \in \mathbb{Z}} c_{f, g} b_{f-p-j} \vec{a}_g \in A(\lambda, -j-1)$$

for all  $j \in \mathbb{Z}$  by Theorem 5.6. This means that,

for each  $g = 0, 1, \dots, n$ ,

$$\sum_{f \in \mathbb{Z}} c_{f, g} b_{f-p-j} = 0 \quad \text{for all } j \in \mathbb{Z} \text{ satisfying } g \geq k(-j) \quad (101)$$

since  $\vec{a}_g \notin A(\lambda, -j-1)$  if  $g \geq k(-j)$  by definition. Here  $k(-j)$  is the number defined by (97) if  $0 \leq -j \leq PHO(\lambda)$  and  $k(-j) = 0$  (resp.  $k(-j) = n+1$ ) if

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$-j < 0$  (resp.  $-j > PHO(\lambda)$ ). Since  $g \geq k(-j)$  is equivalent to  $o(\vec{a}_g, \lambda) \geq -j$  by definition and

$$\sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} = \sum_{f=p+j}^{p+j+q} c_{f,g} b_{f-p-j} = \sum_{s=0}^q c_{p+j+s,g} b_s = 0,$$

the condition (101) is rewritten as the condition

for each  $g = 0, 1, \dots, n$ ,

$$\sum_{s=0}^q c_{p+j+s,g} b_s = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\vec{a}_g, \lambda) \quad (102)$$

Note that coefficients  $b_0$  and  $b_q$  are not zero. Then the condition (102) is equivalent to

$$\begin{aligned} &\text{for each } g = 0, 1, \dots, n, \\ &c_{p+j,g} = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\vec{a}_g, \lambda) \end{aligned} \quad (103)$$

This is just equivalent to the condition (99), which we have to prove.  $\square$

Next we consider  $P(x, \partial)$  of non-zero homogeneous degree.

**Algorithm 8.2** (The case of negative homogeneous degree). *For a given non-zero  $SL_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)^G$  of negative homogeneous degree  $q_1 n < 0$  satisfying the condition*

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (104)$$

*one algorithm to compute a basis of the  $SL_n(\mathbb{R})$ -invariant differential equation  $P(x, \partial)u(x) = 0$  is given in the following.*

**Input:** A non-zero  $SL_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)^G$  of homogeneous degree  $q_1 n < 0$  satisfying the condition (104).

**Output:** A basis of the  $SL_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation  $P(x, \partial)u(x) = 0$ .

**Procedure:**

1. Compute the  $b_P$ -function for  $P(x, \partial)$ . It is denoted by

$$b_P(s) = (s - \lambda_1)^{p_1} \cdots (s - \lambda_l)^{p_l}.$$

2. For each  $\lambda_i$  ( $i = 1, \dots, l$ ), take one standard basis

$$SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \dots, \vec{a}_n(\lambda_i)\}$$

at  $s = \lambda_i$ , which is the standard basis defined by (91) when  $\lambda_i \in \frac{1}{2}\mathbb{Z}$  and the one defined in Definition 5.3 otherwise.

3. Compute the Laurent expansion coefficients

$$\text{Laurent}_{s=\lambda_i}^{(k)}(P[\vec{a}_j(\lambda_i), s](x))$$

for each  $\vec{a}_j(\lambda_i)$  ( $i = 1, \dots, l, j = 0, \dots, n$ ) and  $k \in \mathbb{Z}$  in  $-o_{ij} \leq k \leq -o_{ij}^{\#q_1} + p_i - 1$  with  $o_{ij} := o(\vec{a}_j(\lambda_i), \lambda_i)$  and  $o_{ij}^{\#q_1} := o(\vec{a}_j(\lambda_i)^{\#q_1}, \lambda_i +$

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$q_1$ ). Here,  $o(\vec{a}, \lambda)$  has been defined by (57). Then we have the generators of the vector space  $L_{ij}$  in (105).

$L_{ij} :=$  the vector space generated by

$$\{ \text{Laurent}_{s=\lambda_i}^{(k)} (P[\vec{a}_j(\lambda_i), s](x)) \}_{k=-o_{ij}, \dots, -o_{ij}^{\#q_1} + p_i - 1} \quad (105)$$

Here, if  $-o_{ij} > -o_{ij}^{\#q_1} + p_i - 1$ , then we set  $L_{ij} := \{0\}$ .

4. Then

$$\bigoplus_{\substack{i=1, \dots, l \\ j=0, \dots, n}} L_{ij} \quad (106)$$

forms a basis of the solution space.

**Algorithm 8.3** (The case of positive homogeneous degree). For a given non-zero  $\text{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)^G$  of homogeneous degree  $q_1 n > 0$  satisfying the condition

$$\text{the degree of } b_P(s) = \text{the order of } P(x, \partial), \quad (107)$$

one algorithm to compute a basis of the  $\text{SL}_n(\mathbb{R})$ -invariant differential equation  $P(x, \partial)u(x) = 0$  is given in the following.

**Input:** A non-zero  $\text{SL}_n(\mathbb{R})$ -invariant differential operator  $P(x, \partial) \in D(V)^G$  of homogeneous degree  $q_1 n > 0$  satisfying the condition (107).

**Output:** A basis of the  $\text{SL}_n(\mathbb{R})$ -invariant hyperfunctions to the differential equation  $P(x, \partial)u(x) = 0$ .

**Procedure:**

1. Compute the  $b_P$ -function  $b_P(s)$  and consider the set  $R := R_1 \cup R_2$  with

$$R_1 := \{ \lambda_i := -\frac{i+1}{2} \mid i = 1, 2, \dots, n + 2q_1 - 2 \},$$

$$R_2 := \{ \lambda \in \mathbb{C} \mid b_P(\lambda) = 0 \}.$$

Let  $q_2$  be the number of elements of the set  $R_2 - R_1$ . We denote by

$$\lambda_{n+2q_1-1}, \lambda_{n+2q_1}, \dots, \lambda_{n+2q_1+q_2-2}$$

the elements of  $R_2 - R_1$ . Then we can write the elements of  $R$  by

$$R = \{ \lambda_1, \lambda_2, \dots, \lambda_{n+2q_1+q_2-2} \}.$$

2. We define the multiplicity  $k_i$  of  $\lambda_i$  by

$$p_i := \begin{cases} \text{the multiplicity of } s - \lambda_i \text{ in } b_P(s) & \text{if } b_P(\lambda_i) = 0 \\ 0 & \text{if } b_P(\lambda_i) \neq 0 \end{cases} \quad (108)$$

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3. For each  $\lambda_i$  ( $i = 1, \dots, n + 2q_1 + q_2 - 2$ ), take one standard basis

$$SB^{\lambda_i} = \{\tilde{a}_0(\lambda_i), \dots, \tilde{a}_n(\lambda_i)\}$$

at  $s = \lambda_i$ , which is the standard basis defined by (91) when  $\lambda_i \in \frac{1}{2}\mathbb{Z}$  and the one defined in Definition 5.3 otherwise.

4. Compute the Laurent expansion coefficients

$$Laurent_{s=\lambda_i}^{(k)}(P[\tilde{a}_j(\lambda_i), s](x))$$

for each  $\tilde{a}_j(\lambda_i)$  ( $i = 1, \dots, n + 2q_1 + q_2 - 2, j = 0, \dots, n$ ) and  $k \in \mathbb{Z}$  in  $-o_{ij} \leq k \leq -o_{ij}^{\#q_1} + p_i - 1$  with  $o_{ij} := o(\tilde{a}_j(\lambda_i), \lambda_i)$  and  $o_{ij}^{\#q_1} := o(\tilde{a}_j(\lambda_i), \lambda_i + q_1)$ . Here,  $o(\tilde{a}, \lambda)$  has been defined by (57). Then we have the generators of the vector space  $L_{ij}$  in (109).

$L_{ij} :=$  the vector space generated by

$$\{Laurent_{s=\lambda_i}^{(k)}(P[\tilde{a}_j(\lambda_i), s](x))\}_{k=-o_{ij}, \dots, -o_{ij}^{\#q_1} + p_i - 1} \quad (109)$$

Here, if  $-o_{ij} > -o_{ij}^{\#q_1} + p_i - 1$ , then we set  $L_{ij} := \{0\}$ .

5. Then

$$\bigoplus_{\substack{i=1, \dots, n+2q_1+q_2-2 \\ j=0, \dots, n}} L_{ij} \quad (110)$$

forms a basis of the solution space.

*Proof.* We shall give the proof of Algorithm 8.2 and Algorithm 8.3 simultaneously. First note that we have only to see what  $u(x)$  given as a finite combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$  is annihilated by  $P(x, \partial)$  for each complex number  $\lambda \in \mathbb{C}$  for the same reason in the proof of Algorithm 8.1.

Let

$$SB := \{\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n\}$$

be a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda$  with an increasing sequence

$$0 < k(0) < k(1) < \dots < k(PHO(\lambda)) = n \quad (111)$$

such that

$$SB_q := \{\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{k(q)}\}$$

is a basis of  $A(\lambda, q)$  for each  $q$  in  $0 \leq q \leq PHO(\lambda)$ . In particular, we suppose that it is the standard basis defined by (91) when  $\lambda_i \in \frac{1}{2}\mathbb{Z}$  and the one defined in Definition 5.3 otherwise. Then, by the property (93), we see easily that

$$SB^{\#q_1} := \{\tilde{a}_0^{\#q_1}, \tilde{a}_1^{\#q_1}, \dots, \tilde{a}_n^{\#q_1}\}$$

is a standard basis of  $\mathbb{C}^{n+1}$  at  $s = \lambda + q_1$  with an increasing sequence

$$0 < k^{\#q_1}(0) < k^{\#q_1}(1) < \dots < k^{\#q_1}(PHO(\lambda + q_1)) = n \quad (112)$$

such that

$$SB_q^{\#q_1} := \{\bar{a}_0^{\#q_1}, \bar{a}_1^{\#q_1}, \dots, \bar{a}_{k^{\#q_1}(q)}^{\#q_1}\}$$

is a basis of  $A(\lambda + q_1, q)$  for each  $q$  in  $0 \leq q \leq PHO(\lambda + q_1)$ . Here, we see from the definition that

$$\begin{aligned} PHO(\lambda + q_1) &\geq PHO(\lambda) \quad \text{if } q_1 < 0 \\ PHO(\lambda + q_1) &\leq PHO(\lambda) \quad \text{if } q_1 > 0 \end{aligned}$$

and that

$$k(q) = k^{\#q_1}(q)$$

for  $q < PHO(\lambda)$  if  $q_1 < 0$  or for  $q < PHO(\lambda + q_1)$  if  $q_1 > 0$ .

Every  $u(x)$  given as a finite combination of Laurent expansion coefficients of  $|P(x)|_i^s$  ( $i = 0, \dots, n$ ) at  $s = \lambda$  is expressed as a linear combination

$$u(x) = \sum_{\substack{f, g \in \mathbb{Z} \\ 0 \leq g \leq n}} c_{f,g} P_f^{[\bar{a}_g, \lambda]}(x) \quad (113)$$

with

$$P_f^{[\bar{a}_g, \lambda]}(x) = \text{Laurent}_{s=\lambda}^{(f)}(P^{[\bar{a}_g, s]}(x))$$

and  $c_{f,g} \in \mathbb{C}$ . Let

$$b_P(s) = \sum_{i=0}^q b_i (s - \lambda)^{p+i}$$

be a expansion of  $b_P(s)$  with respect to  $(s - \lambda)$ . The number  $p$  is the multiplicity of  $b_P(s)$  at  $s = \lambda$ .

Then *what we have to prove is that*

$$\begin{aligned} c_{f,g} &= 0 \text{ except for } -o(\bar{a}_g, \lambda) \leq f \leq -o(\bar{a}_g^{\#q_1}, \lambda + q_1) + p - 1 \\ &\text{if and only if } P(x, \partial)u(x) = 0 \end{aligned} \quad (114)$$

since  $P_f^{[\bar{a}_g, \lambda]}(x) = 0$  if  $f < -o(\bar{a}_g, \lambda)$  from the definition. Here,  $o(\bar{a}, \lambda)$  has been defined by (57).

Indeed, first we consider the situation that  $\lambda$  is not a root of  $b_P(s) = 0$ , i.e.,  $p = 0$ . When  $q_1 < 0$  (Algorithm 8.2), there is no non-zero  $G$ -invariant homogeneous solutions of homogeneous degree  $n\lambda$  to  $P(x, \partial)u(x) = 0$ . When  $q_1 > 0$  (Algorithm 8.3), there is no non-zero  $G$ -invariant homogeneous solutions of homogeneous degree  $n\lambda$  to  $P(x, \partial)u(x) = 0$  *except that*  $\lambda \in R_1$ . Then we have only to consider the cases that  $\lambda$  is a root of  $b_P(\lambda) = 0$  when  $q_1 < 0$  (Algorithm 8.2), and the cases  $\lambda$  is a root of  $b_P(\lambda) = 0$  or  $\lambda \in R_1$  when  $q_1 > 0$  (Algorithm 8.3). This is the reason why we restrict the  $\lambda$ 's to the finite sets of numbers in the first step of the procedures in the algorithms. We can easily see that the basis of  $L_{ij}$  in (109) is just the basis of the terms in the expression (113) with the condition (114) when  $\lambda = \lambda_i$  and

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The rest of the proof is devoted to proving (114). The Laurent expansion of  $P^{[\vec{a},s]}(x)$  at  $s = \lambda$  is denoted by

$$P^{[\vec{a},s]}(x) = \sum_{w \in \mathbb{Z}} P_w^{[\vec{a},\lambda]}(x)(s - \lambda)^w.$$

Then we have

$$\begin{aligned} P(x, \partial)P^{[\vec{a},s]}(x) &= \sum_{w \in \mathbb{Z}} P(x, \partial)P_w^{[\vec{a},\lambda]}(x)(s - \lambda)^w = b_P(s)P^{[\vec{a}^{\#q_1}, s - q_1]}(x) \\ &= \left( \sum_{i=0}^q b_i(s - \lambda)^{p+i} \right) \left( \sum_{j \in \mathbb{Z}} P_j^{[\vec{a}^{\#q_1}, \lambda + q_1]}(x)(s - \lambda)^j \right) \\ &= \sum_{w \in \mathbb{Z}} \sum_{i+j+p=w} b_i P_j^{[\vec{a}^{\#q_1}, \lambda + q_1]}(x)(s - \lambda)^w \end{aligned}$$

and hence we have

$$P(x, \partial)P_w^{[\vec{a},\lambda]}(x) = \sum_{i+j+p=w} b_i P_j^{[\vec{a}^{\#q_1}, \lambda + q_1]}(x). \quad (115)$$

Here  $b_i = 0$  except for  $i$  in  $0 \leq i \leq q$  and  $P_j^{[\vec{a}^{\#q_1}, \lambda + q_1]}(x) = 0$  for sufficiently small  $j$ . Then for

$$u(x) = \sum_{\substack{f, g \in \mathbb{Z} \\ 0 \leq g \leq n}} c_{f,g} P_f^{[\vec{a}_g, \lambda]}(x),$$

we have

$$\begin{aligned} P(x, \partial)u(x) &= \sum_{f, g \in \mathbb{Z}} c_{f,g} P(x, \partial)P_f^{[\vec{a}_g, \lambda]}(x) = \sum_{f, g \in \mathbb{Z}} c_{f,g} \sum_{i=0}^q b_i P_{f-p-i}^{[\vec{a}_g^{\#q_1}, \lambda + q_1]}(x) \\ &= \sum_{f, g \in \mathbb{Z}} c_{f,g} \sum_{j \in \mathbb{Z}} b_{f-p-j} P_j^{[\vec{a}_g^{\#q_1}, \lambda + q_1]}(x) = \sum_{j \in \mathbb{Z}} \sum_{f, g \in \mathbb{Z}} c_{f,g} b_{f-p-j} P_j^{[\vec{a}_g^{\#q_1}, \lambda + q_1]}(x) \\ &= \sum_{j \in \mathbb{Z}} P_j^{[\sum_{f, g \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g^{\#q_1}, \lambda + q_1]}(x) = 0 \end{aligned}$$

where  $g$  runs in  $0 \leq g \leq n$ , i.e.,  $\vec{a}_g = 0$  except for  $0 \leq g \leq n$ . Then we have

$$\sum_{f, g \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g = \sum_{g=0}^n \sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} \vec{a}_g^{\#q_1} \in A(\lambda + q_1, -j - 1)$$

for all  $j \in \mathbb{Z}$  by Theorem 5.6. This means that,

for each  $g = 0, 1, \dots, n$ ,

$$\sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} = 0 \quad \text{for all } j \in \mathbb{Z} \text{ satisfying } g \geq k^{\#q_1}(-j) \quad (116)$$

since  $\vec{a}_g^{\#q_1} \notin A(\lambda + q_1, -j - 1)$  if  $g \geq k^{\#q_1}(-j)$  by definition. Here  $k^{\#q_1}(-j)$  is the number defined by (112) if  $0 \leq -j \leq PHO(\lambda + q_1)$  and  $k^{\#q_1}(-j) = 0$



(resp.  $k^{\#q_1}(-j) = n + 1$ ) if  $-j < 0$  (resp.  $-j > PHO(\lambda + q_1)$ ). Since  $g \geq k^{\#q_1}(-j)$  is equivalent to  $o(\tilde{a}_g^{\#q_1}, \lambda + q_1) \geq -j$  by definition and

$$\sum_{f \in \mathbb{Z}} c_{f,g} b_{f-p-j} = \sum_{f=p+j}^{p+j+q} c_{f,g} b_{f-p-j} = \sum_{s=0}^q c_{p+j+s,g} b_s = 0,$$

the condition (116) is rewritten as the condition

for each  $g = 0, 1, \dots, n$ ,

$$\sum_{s=0}^q c_{p+j+s,g} b_s = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\tilde{a}_g^{\#q_1}, \lambda + q_1) \quad (117)$$

Note that coefficients  $b_0$  and  $b_q$  are not zero. Then the condition (117) is equivalent to

$$\begin{aligned} &\text{for each } g = 0, 1, \dots, n, \\ &c_{p+j,g} = 0 \text{ for all } j \in \mathbb{Z} \text{ satisfying } j \geq -o(\tilde{a}_g^{\#q_1}, \lambda + q_1) \end{aligned} \quad (118)$$

This is just equivalent to the condition (114), which we have to prove.  $\square$

## 9. ALGORITHMS FOR CONSTRUCTING SOLUTIONS — INHOMOGENEOUS EQUATIONS.

**Algorithm 9.1** (The case of inhomogeneous equation). *For a given non-zero  $\mathrm{SL}_n(\mathbb{R})$ -invariant homogeneous differential operator  $P(x, \partial) \in D(V)^G$  of homogeneous degree  $kn \in \mathbb{Z}$  satisfying the condition*

$$b_P(s) \not\equiv 0 \quad (119)$$

*one algorithm to compute a  $G$ -invariant hyperfunction solution of the  $\mathrm{SL}_n(\mathbb{R})$ -invariant differential equation  $P(x, \partial)u(x) = v(x)$  is given in the following.*

**Input:** *A non-zero  $\mathrm{SL}_n(\mathbb{R})$ -invariant homogeneous differential operator  $P(x, \partial) \in D(V)^G$  of homogeneous degree  $kn \in \mathbb{Z}$  satisfying the condition (119) and a non-zero quasi-homogeneous  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction  $v(x)$ .*

**Output:** *A non-zero  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction  $u(x)$  to the differential equation  $P(x, \partial)u(x) = v(x)$ .*

**Procedure:**

1. *Write  $v(x)$  as a sum of Laurent expansion coefficients. Namely,  $v(x)$  is given by*

$$v(x) = \sum_{l=1}^q P_{w_l}^{[\tilde{a}_l, \lambda]}(x)$$

*with  $\tilde{a}_1, \dots, \tilde{a}_k \in \mathbb{C}^{n+1}$  and  $w_1, \dots, w_k \in \mathbb{Z}$ .*

2. *Compute the  $b_P$ -function  $b_P(s)$  and divide it as*

$$b_P(s) = (s - \lambda + k)^p \tilde{b}(s), \quad (\tilde{b}(\lambda - k) \neq 0)$$

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and expand  $\tilde{b}(s)^{-1}$  into a Taylor series at  $s = \lambda - k$ ,

$$\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i (s - \lambda + k)^i$$

3. Then

$$u(x) = \sum_{l=1}^q \sum_{i+j=w_l+p} b_i P_j^{[\tilde{\alpha}_l^{\#-k}, \lambda-k]}(x) \quad (120)$$

is an  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solution to  $P(x, \partial)u(x) = v(x)$ .

*Proof.* The proof can be found in the proof of Theorem 6.1.  $\square$

## 10. EXPLICIT COMPUTATIONS OF EXAMPLES

We shall give in this section some examples. Some homogeneous differential equations generated by  $\det(x)$  and  $\det(\partial^*)$  are dealt with in this section.

**10.1. The equations  $\det(\partial^*) \det(x)u(x) = 0$  and  $\det(x) \det(\partial^*)u(x) = 0$ .** First we consider two examples of differential equation of homogeneous degree 0. Let us consider the case of  $P(x, \partial) = \det(\partial^*) \det(x)$  and  $P(x, \partial) = \det(x) \det(\partial^*)$ . The homogeneous degrees of  $P(x, \partial)$  are 0 and the  $b_P$ -functions are  $b_P(s) = (s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2})$  and  $b_P(s) = (s)(s+\frac{1}{2})\cdots(s+\frac{n-1}{2})$ , respectively.

**Proposition 10.1.** *First we consider the differential equations  $\det(\partial^*) \det(x)u(x) = 0$  and  $\det(x) \det(\partial^*)u(x) = 0$ .*

1. *The  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(\partial^*) \det(x)u(x) = 0$  is generated by*

$$\bigcup_{i=1}^n \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-q)}(P^{[\tilde{\alpha}, s]}(x)) \mid \tilde{\alpha} \in A(-\frac{i+1}{2}, q) \right\} \quad (121)$$

Here,  $A(-\frac{i+1}{2}, q)$  is a vector subspace of  $\mathbb{C}^{n+1}$  defined by (54) in Definition 5.1 and explicitly computed in (88) of Theorem 7.1. Similarly, the  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(x) \det(\partial^*)u(x) = 0$  is generated by

$$\bigcup_{i=-1}^{n-2} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-q)}(P^{[\tilde{\alpha}, s]}(x)) \mid \tilde{\alpha} \in A(-\frac{i+1}{2}, q) \right\} \quad (122)$$

2. *In particular, for  $i = -1, 0, 1, 2, \dots, n$ ,*

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-q)}(P^{[\tilde{\alpha}, s]}(x)) \mid \tilde{\alpha} \in A(-\frac{i+1}{2}, q) \right\} \quad (123)$$

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forms an  $n + 1$ -dimensional vector space generated by all the relatively invariant hyperfunctions under the action of  $g \in \mathrm{GL}_n(\mathbb{R})$  corresponding to the character  $\det(g)^{-i-1}$ . The dimensions of  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solutions to  $\det(\partial^*) \det(x) u(x) = 0$  and  $\det(x) \det(\partial^*) u(x) = 0$  are  $n(n + 1)$ .

*Proof.* 1. We compute the solution space following Algorithm 8.1. For the differential operator  $P(x, \partial) = \det(\partial^*) \det(x)$ , the  $b_P$ -function is

$$b_P(s) = (s + 1)(s + \frac{3}{2}) \cdots (s + \frac{n+1}{2}).$$

In the first step in the procedure of Algorithm 8.1, we have  $l = n$  and the roots of  $b_P(s) = 0$  are  $\lambda_i = \frac{i+1}{2}$  with multiplicity  $p_i = 1$  ( $i = 1, \dots, n$ ). Since they are all half-integers, we can take a standard basis at  $s = \lambda_i$

$$SB^{\lambda_i} = \{\bar{a}_0(\lambda_i), \dots, \bar{a}_n(\lambda_i)\}$$

as the one defined in Definition 7.3. Let  $SB_q^{\lambda_i}$  be a subset of  $SB^{\lambda_i}$  such that  $SB_q^{\lambda_i}$  forms a basis of  $A(\lambda_i, q)$  for each  $q$  in  $0 \leq q \leq \mathrm{PHO}(\lambda_i) = \lfloor \frac{i+1}{2} \rfloor$ . Then we have

$$SB^{\lambda_i} = \bigsqcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}.$$

and the set  $SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$  forms a basis of  $\overline{A(\lambda_i, q)} := A(\lambda_i, q)/A(\lambda_i, q-1)$ , where  $A(\lambda_i, -1) = \{0\}$  and  $SB_{-1}^{\lambda_i} := \emptyset$ . For each  $\bar{a}_j(\lambda_i) \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$ , we have  $o_{ij} := o(\bar{a}_j(\lambda_i), \lambda_i) = q$  and hence

$$-o_{ij} \leq k \leq -o_{ij} + p_i - 1 \Rightarrow -q \leq k \leq -q + p_i - 1 \Rightarrow k = -q$$

for each  $q$  in  $0 \leq q \leq \lfloor \frac{i+1}{2} \rfloor$ . Since

$$Laurent_{s=\lambda_i}^{(-q)}(P^{[\bar{a}, s]}(x)) = 0$$

if  $\bar{a} \in A(\lambda_i, q-1)$ , we have

$$\begin{aligned} & \text{Vector space generated by } \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\bar{a}, s]}(x)) \mid \bar{a} \in A(\lambda_i, q)\} \\ &= \text{Vector space generated by } \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\bar{a}, s]}(x)) \mid [\bar{a}] \in \overline{A(\lambda_i, q)}\} \\ &= \text{Vector space generated by } \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\bar{a}, s]}(x)) \mid \bar{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}\}, \end{aligned}$$

for each  $q$  in  $0 \leq q \leq \lfloor \frac{i+1}{2} \rfloor$ . Then the vector spaces generated by

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\bar{a}, s]}(x)) \mid \bar{a} \in A(\lambda_i, q)\}$$

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and

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\vec{a},s]}(x))\} = \bigoplus_{j=0}^n L_{ij}$$

coincide with each other, and so the vector spaces generated by

$$\bigcup_{i=1}^n \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\vec{a},s]}(x)) \mid \vec{a} \in A(\lambda_i, q)\} \quad (124)$$

and the vector space generated by

$$\bigoplus_{i=1}^n \bigoplus_{j=0}^n L_{ij} = \bigcup_{i=1}^n \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\vec{a},s]}(x))\}.$$

Thus we have proved that (121), which is (124), forms a basis of the vector space of  $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(\partial^*) \det(x) u(x) = 0$ .

For the differential equation  $\det(x) \det(\partial^*) u(x) = 0$ , we can prove in the same way that the vector space of  $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(x) \det(\partial^*) u(x) = 0$  is generated by (122) since the  $b_P$ -function of  $P(x, \partial) = \det(x) \det(\partial^*)$  is

$$b_P(s) = (s)(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2}).$$

2. By Proposition 5.5, the elements in

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \bigcup_{\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}} \{Laurent_{s=\lambda_i}^{(-q)}(P^{[\vec{a},s]}(x))\} \quad (125)$$

are linearly independent and forms an  $n+1$ -dimensional vector space. On the other hand, by Theorem 5.6, each

$$Laurent_{s=\lambda_i}^{(-q)}(P^{[\vec{a},s]}(x))$$

for  $\vec{a} \in SB_q^{\lambda_i} - SB_{q-1}^{\lambda_i}$  is a homogeneous  $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunction of homogeneous degree  $n\lambda_i = -n(i+1)/2$ . This means that it is relatively invariant under the action of  $\mathbf{GL}_n(\mathbb{R})$  corresponding to the character  $\det(g)^{-i-1}$ . By the main result of [11], the space of relatively invariant hyperfunctions for a fixed character  $\det(g)^{2s}$  ( $s \in \mathbb{C}$ ) is  $n+1$ . Then (125) forms a basis of all relatively invariant hyperfunctions corresponding to the character  $\det(g)^{-i-1}$ . Then we see that the dimensions of  $\mathbf{SL}_n(\mathbb{R})$ -invariant hyperfunction solutions to  $\det(\partial^*) \det(x) u(x) = 0$  and  $\det(x) \det(\partial^*) u(x) = 0$  are  $n(n+1)$ .

**10.2. The equations  $\det(x)u(x) = 0$ .** Let us consider the case of  $P(x, \partial) = \det(x)$ . Then the total homogeneous degree of  $P(x, \partial)$  is  $n$  and  $b_P(s) = 1$ . We can prove by our algorithm that the  $G$ -invariant solution space of the differential equation  $\det(x)u(x) = 0$  is generated by the  $G$ -invariant measures on all the singular orbits (i.e.,  $G$ -orbits contained in  $\det(x) = 0$ ), and hence, it is  $\frac{n(n+1)}{2}$ -dimensional (= the number of singular orbits). Here the  $G$ -invariant measure on each singular orbit is a relatively invariant hyperfunction. Namely we have the following proposition.

**Proposition 10.2.** *Consider the differential equation  $\det(x)u(x) = 0$ .*

1. *The  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(x)u(x) = 0$  is generated by*

$$\bigcup_{i=1}^n \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-\lfloor \frac{i+1}{2} \rfloor)}(P[\vec{a}, s](x)) \mid \vec{a} \in \mathbb{C}^{n+1} \right\} \quad (126)$$

2. *In particular, for  $i = 1, 2, \dots, n$ ,*

$$\left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-\lfloor \frac{i+1}{2} \rfloor)}(P[\vec{a}, s](x)) \mid \vec{a} \in \mathbb{C}^{n+1} \right\} \quad (127)$$

*forms an  $(n+1-i)$ -dimensional vector space generated by the tempered distributions*

$$f(x) \longmapsto \int f(x) d\nu_i^j \quad (f(x) \in \mathcal{S}(V))$$

*( $j = 0, 1, \dots, n-i$ ) where  $d\nu_i^j$  is the  $\mathrm{SL}_n(\mathbb{R})$ -invariant measure on*

$$S_i^j := \{x \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{sgn}(x) = (j, n-i-j)\}$$

*Proof.* 1. We shall prove it by carrying out Algorithm 8.3.

The  $b_P$ -function is  $b_P(s) \equiv 1$  and we have

$$R = R_1 = \{\lambda_i = -\frac{i+1}{2} \mid i = 1, \dots, n\}$$

and  $q_1 = 1$  and  $q_2 = 0$  in the first step of the procedure. In the second step, we have  $p_i = 0$  for all  $i = 1, \dots, n$ . Since all  $\lambda_i \in R$  are half-integers, we take the standard basis

$$SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \dots, \vec{a}_n(\lambda_i)\}$$

defined by Definition 5.3 as a standard basis at  $s = \lambda_i$ . For each  $\vec{a}_j(\lambda_i)$  ( $i = 1, \dots, n$  and  $j = 0, \dots, n$ ), we have defined

$$\begin{aligned} o_{ij} &:= o(\vec{a}_j(\lambda_i), \lambda_i), \\ o_{ij}^{\#q_1} &:= o_{ij}^{\#} = o(\vec{a}_j(\lambda_i)^{\#}, \lambda_i + 1). \end{aligned}$$

Let  $SB_{q-1}^{\lambda_i}$  be a subset of  $SB^{\lambda_i}$  consisting of the vectors  $\vec{a}_j(\lambda_i)$  such that  $P[\vec{a}_j(\lambda_i), s](x)$  has a pole of order less than  $q-1$  at  $s = \lambda_i$ . If  $\vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB_{q-1}^{\lambda_i}$ , then  $P[\vec{a}_j(\lambda_i), s](x)$  has a pole of order  $q$  at  $s = \lambda_i$ .

since the possible highest order of  $P^{[\vec{a},s]}(x)$  at  $s = \lambda_i$  is  $q = \lfloor \frac{i+1}{2} \rfloor$ . Then we have, by putting  $q = \lfloor \frac{i+1}{2} \rfloor$ ,

$$\begin{aligned} o_{ij} &= o_{ij}^{\#} + 1 = q \quad \text{if } \vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB_{q-1}^{\lambda_i}, \\ o_{ij} &= o_{ij}^{\#} \leq q - 1 \quad \text{if } \vec{a}_j(\lambda_i) \in SB_{q-1}^{\lambda_i}. \end{aligned} \quad (128)$$

Indeed, by Theorem 7.1, Definition 7.3 and the property (93), we see that

$$\begin{aligned} &(\text{the order of } P^{[\vec{a}_j(\lambda_i)^{\#},s]}(x) \text{ at } s = \lambda_i + 1) \\ &= (\text{the order of } P^{[\vec{a}_j(\lambda_i),s]}(x) \text{ at } s = \lambda_i) \end{aligned}$$

for all  $\vec{a}_j(\lambda_i) \in SB_{q-1}^{\lambda_i}$  and that

$$(\text{the order of } P^{[\vec{a}_j(\lambda_i)^{\#},s]}(x) \text{ at } s = \lambda_i + 1) = q - 1$$

for all  $\vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB_{q-1}^{\lambda_i}$ . Then we have (128). Thus, for  $\vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB_{q-1}^{\lambda_i}$ , we have

$$-o_{ij} = -(o_{ij}^{\#} + 1) = -o_{ij}^{\#} + p_i - 1 = -q = -\lfloor \frac{i+1}{2} \rfloor,$$

and  $L_{ij}$  in (109) is generated by

$$Laurent_{s=\lambda_i}^{-\lfloor \frac{i+1}{2} \rfloor} (P^{[\vec{a}_j(\lambda_i),s]}(x)).$$

For  $\vec{a}_j(\lambda_i) \in SB_{q-1}^{\lambda_i}$ , we have

$$-o_{ij} = -o_{ij}^{\#} > -o_{ij} - 1 = -o_{ij} + p_i - 1,$$

and hence  $L_{ij}$  in (109) is  $\{0\}$ . Therefore we have

$$\begin{aligned} &\bigoplus_{\substack{i=1,\dots,n \\ j=0,\dots,n}} L_{ij} \\ &= \bigoplus_{i=1,\dots,n} \left( \begin{array}{l} \text{the vector space generated by} \\ \{Laurent_{s=\lambda_i}^{-\lfloor \frac{i+1}{2} \rfloor} (P^{[\vec{a}_j(\lambda_i),s]}(x)) \mid \vec{a}_j(\lambda_i) \in SB^{\lambda_i} - SB_{q-1}^{\lambda_i}\} \end{array} \right) \\ &= \left( \begin{array}{l} \text{the vector space generated by} \\ \bigcup_{i=1}^n \{Laurent_{s=-\frac{i+1}{2}}^{-\lfloor \frac{i+1}{2} \rfloor} (P^{[\vec{a},s]}(x)) \mid \vec{a} \in \mathbb{C}^{n+1}\} \end{array} \right), \end{aligned}$$

since  $Laurent_{s=-\frac{i+1}{2}}^{-\lfloor \frac{i+1}{2} \rfloor} (P^{[\vec{a}_j(\lambda_i),s]}(x)) = 0$  if  $\vec{a}_j(\lambda_i) \in SB_{q-1}^{\lambda_i}$ . This is what we want to prove.

2. For  $i = 1, \dots, n$ , each element of

$$\{Laurent_{s=-\frac{i+1}{2}}^{-\lfloor \frac{i+1}{2} \rfloor} (P^{[\vec{a},s]}(x)) \mid \vec{a} \in \mathbb{C}^{n+1}\}$$

is a homogeneous  $SL_n(\mathbb{R})$ -invariant hyperfunction of homogeneous degree  $-n(\frac{i+1}{2})$  and its support is contained in  $S_i$  (Theorem A.2). It is

proved that such hyperfunctions are given as a linear sum of  $\mathrm{SL}_n(\mathbb{R})$ -invariant measures on the  $(n-i+1)$  open orbits  $S_i^j$  ( $j = 1, \dots, n-i+1$ ) in  $S_i$ . See, for example, §4 in [9]. Thus we have the result.  $\square$

**10.3. The equations**  $\det(\partial^*)u(x) = 0$ . Similar argument is possible for the case of  $P(x, \partial) = \det(\partial)$ . In this case, the total homogeneous degree of  $P(x, \partial)$  is  $(-n)$  and we see that  $b_P(s) = \prod_{i=1}^n (s + \frac{i-1}{2})$ . The solution space of  $\det(\partial)u(x) = 0$  is just the Fourier transform of that of  $\det(x)u(x) = 0$ , and hence it is  $\frac{n(n+1)}{2}$ -dimensional and generated by relatively invariant hyperfunctions. We can construct them from the complex power of  $\det(x)$

**Proposition 10.3.** *Consider the differential equation  $\det(\partial^*)u(x) = 0$ .*

1. *The  $\mathrm{SL}_n(\mathbb{R})$ -invariant hyperfunction solution space to the differential equation  $\det(\partial^*)u(x) = 0$  is generated by*

$$\bigcup_{i=-1}^{n-2} \bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-q)}(P^{[\vec{a}, s]}((x))) \mid \vec{a} \in D_*^{(q)} \right\} \quad (129)$$

Here,  $D_*^{(j)}$  is a vector subspace of  $\mathbb{C}^{n+1}$  defined by Definition 7.2. The  $*$  in  $D_*^{(j)}$  is substituted half, even or odd according as  $-\frac{i+1}{2}$  is a strictly half integer, an even integer or an odd integer, respectively.

2. *In particular, for  $i = -1, 0, 1, \dots, n-2$ ,*

$$\bigcup_{q=0}^{\lfloor \frac{i+1}{2} \rfloor} \left\{ \text{Laurent}_{s=-\frac{i+1}{2}}^{(-q)}(P^{[\vec{a}, s]}((x))) \mid \vec{a} \in D_*^{(q)} \right\} \quad (130)$$

*forms an  $(i+2)$ -dimensional vector space generated by the Fourier transforms of the tempered distributions in (127).*

*Proof.* We follow Algorithm 8.2. The first step and the second step of the procedure of Algorithm 8.2 are the same as those of Algorithm 8.1. The roots of the  $b_P$ -function are

$$\lambda_i = -\frac{i+1}{2} \quad (i = -1, 0, \dots, n-2)$$

and their multiplicity  $p_i$  is 1. We can determine the generators of the solution space in the same way as the proof of Proposition 10.1. Since they are all half-integers, we can take a standard basis at  $s = \lambda_i$

$$SB^{\lambda_i} = \{\vec{a}_0(\lambda_i), \dots, \vec{a}_n(\lambda_i)\}$$

as the one defined in Definition 7.3. For each  $\vec{a}_j(\lambda_i) \in SB^{\lambda_i}$ , we define

$$o_{ij} := o(\vec{a}_j(\lambda_i), \lambda_i)$$

and

$$o_{ij}^{\#q_1} := o_{ij}^{\#-1} = o(\vec{a}_j(\lambda_i)^{\#-1}, \lambda_i - 1).$$

We have only to pick up the vector  $\vec{a}_j(\lambda_i)$  satisfying

$$-o_{ij} \leq -o_{ij}^{\#-1} + p_i - 1 = -o_{ij}^{\#-1}. \quad (131)$$

Since for  $i = -1, 0, \dots, n-2$ , we see  $-o_{ij} \geq -o_{ij}^{\#-1}$  by Theorem 7.1, Definition 7.3 and Theorem A.1, (131) means

$$-o_{ij} = -o_{ij}^{\#-1}. \quad (132)$$

Namely, we have to choose  $\vec{a}_j(\lambda_i)$  satisfying

$$\begin{aligned} & \text{(the order of } P^{[\vec{a}_j(\lambda_i)^{\#-1}, s]}(x) \text{ at } s = \lambda_i - 1) \\ & = \text{(the order of } P^{[\vec{a}_j(\lambda_i), s]}(x) \text{ at } s = \lambda_i). \end{aligned}$$

By Theorem 7.1, Definition 7.3 and Theorem A.1, we see that this condition is equivalent to that

$$\vec{a}_j(\lambda_i) \in D_*^{(q)}$$

with some  $q = 0, 1, \dots, \lfloor \frac{i+1}{2} \rfloor$ . Here, the  $*$  in  $D_*^{(j)}$  is substituted *half, even or odd* according as  $-\frac{i+1}{2}$  is a strictly half integer, an even integer or an odd integer, respectively. Thus we have the first result.

The second result is easily verified.  $\square$

**10.4. The equations**  $P(x, \partial)u(x) = \delta(x)$ . We shall find a  $G$ -invariant fundamental solution to the homogeneous  $G$ -invariant differential operator  $P(x, \partial)$ . First note that the delta function  $\delta(x)$  on  $\text{Sym}_n(\mathbb{R})$  is given as

$$\begin{aligned} \delta(x) &= (\text{const.}) \times P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\vec{a}, -\frac{n+1}{2}]}(x) \\ &= (\text{const.}) \times \text{Laurent}_{s=-\frac{n+1}{2}}^{(-\lfloor \frac{n+1}{2} \rfloor)}(P^{[\vec{a}, s]}(x)) \end{aligned}$$

with a vector  $\vec{a} \in A(-\frac{n+1}{2}, \lfloor \frac{n+1}{2} \rfloor)$  which is non-zero in  $\overline{A(-\frac{n+1}{2}, \lfloor \frac{n+1}{2} \rfloor)}$ . Henceforth, we fix  $\vec{a} = \vec{a}_0$  satisfying

$$\delta(x) = P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\vec{a}_0, -\frac{n+1}{2}]}(x). \quad (133)$$

**Example 10.1.** We consider the differential operator  $P(x, \partial) = \det(\partial^*) \det(x)$ . It is a operator of homogeneous degree 0 and hence  $k = 0$  in Algorithm 9.1. The  $b_P$ -function is

$$b_P(s) = (s+1)(s+\frac{3}{2}) \dots (s+\frac{n+1}{2}).$$

The function  $v(x)$  in Algorithm 9.1 is given by (133) and hence  $\lambda = -\frac{n+1}{2}$  in Algorithm 9.1. Since  $(s - \lambda + k) = (s + \frac{n+1}{2})$ , we have

$$b_P(s) = (s + \frac{n+1}{2}) \tilde{b}(s)$$

where

$$\tilde{b}(s) = (s+1)(s+\frac{3}{2}) \dots (s+\frac{n}{2}).$$



Then  $p = 1$  in Algorithm 9.1. We have the Taylor expansion

$$\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i \left(s + \frac{n-1}{2}\right)^i$$

with

$$b_0 = \left( \left(-\frac{n-1}{2}\right) \left(-\frac{n-2}{2}\right) \cdots \left(-\frac{1}{2}\right) \right)^{-1}.$$

Then, by the third step of the procedure in Algorithm 9.1, we have

$$u(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor + 1} b_i P_j^{[\tilde{a}_0, -\frac{n+1}{2}]}(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor} b_i P_j^{[\tilde{a}_0, -\frac{n+1}{2}]}(x)$$

Since  $i \geq 0$  and  $P_j^{[\tilde{a}_0, -\frac{n+1}{2}]}(x) = 0$  except for  $j \geq -\lfloor \frac{n+1}{2} \rfloor$ , we have

$$u(x) = b_0 \times P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\tilde{a}_0, -\frac{n+1}{2}]}(x) + b_1 \times P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\tilde{a}_0, -\frac{n+1}{2}]}(x).$$

This is a  $G$ -invariant fundamental solution to  $P(x, \partial) = \det(\partial^*) \det(x)$ . In this case,  $b_1 \times P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\tilde{a}_0, -\frac{n+1}{2}]}(x)$  is not necessary since it is annihilated by  $P(x, \partial)$ .

Next we consider the differential operator  $P(x, \partial) = \det(x) \det(\partial^*)$ . It is a operator of homogeneous degree 0 and hence  $k = 0$  in Algorithm 9.1. The  $b_P$ -function is

$$b_P(s) = (s)(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2}).$$

The function  $v(x)$  in Algorithm 9.1 is given by (133) and hence  $\lambda = -\frac{n+1}{2}$  in Algorithm 9.1. Since  $(s - \lambda + k) = (s + \frac{n+1}{2})$ , we have

$$b_P(s) = \tilde{b}(s) = (s)(s + \frac{1}{2}) \cdots (s + \frac{n-1}{2}).$$

Then  $p = 0$  in Algorithm 9.1. We have the Taylor expansion

$$\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i \left(s + \frac{n-1}{2}\right)^i$$

with

$$b_0 = \left( \left(-\frac{n+1}{2}\right) \left(-\frac{n}{2}\right) \cdots (-1) \right)^{-1}.$$

Then, by the third step of the procedure in Algorithm 9.1, we have

$$u(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor} b_i P_j^{[\tilde{a}_0, -\frac{n+1}{2}]}(x)$$

Since  $i \geq 0$  and  $P_j^{[\tilde{a}_0, -\frac{n+1}{2}]}(x) = 0$  except for  $j \geq -\lfloor \frac{n+1}{2} \rfloor$ , we have

$$u(x) = \left( \left(-\frac{n+1}{2}\right) \left(-\frac{n}{2}\right) \cdots (-1) \right)^{-1} \times P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\tilde{a}_0, -\frac{n+1}{2}]}(x)$$

This is a  $G$ -invariant fundamental solution of  $P(x, \partial) = \det(x) \det(\partial^*)$ .

**Example 10.2.** We consider the differential operator  $P(x, \partial) = \det(x)$ . It is a operator of homogeneous degree  $n$  and hence  $k = 1$  in Algorithm 9.1. The  $b_P$ -function is

$$b_P(s) = 1$$

The function  $v(x)$  in Algorithm 9.1 is given by (133) and hence  $\lambda = -\frac{n+1}{2}$  in Algorithm 9.1. Since  $(s - \lambda + k) = (s + \frac{n+3}{2})$ , we have  $p = 0$  in Algorithm 9.1 and  $\tilde{b}(s)^{-1} = 1$ , and hence  $b_0 = 1$  and  $b_i = 0$  for  $i > 1$ . Then, by the third step of the procedure in Algorithm 9.1, we have

$$u(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor + 0} b_i P_j^{[\tilde{\alpha}_0^{\#1}, -\frac{n+1}{2}-1]}(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor} b_i P_j^{[\tilde{\alpha}_0^{\#1}, -\frac{n+3}{2}]}(x)$$

Since  $i \geq 0$  and  $P_j^{[\tilde{\alpha}_0^{\#1}, -\frac{n+3}{2}]}(x) = 0$  except for  $j \geq -\lfloor \frac{n+1}{2} \rfloor$ , we have

$$u(x) = P_{-\lfloor \frac{n+1}{2} \rfloor}^{[\tilde{\alpha}_0^{\#1}, -\frac{n+3}{2}]}(x).$$

This is a  $G$ -invariant fundamental solution of  $P(x, \partial) = \det(x)$ .

**Example 10.3.** We consider the differential operator  $P(x, \partial) = \det(\partial^*)$ . It is a operator of homogeneous degree  $-n$  and hence  $k = -1$  in Algorithm 9.1. The  $b_P$ -function is

$$b_P(s) = s(s + \frac{1}{2}) \dots (s + \frac{n-1}{2}).$$

The function  $v(x)$  in Algorithm 9.1 is given by (133) and hence  $\lambda = -\frac{n+1}{2}$  in Algorithm 9.1. Since  $(s - \lambda + k) = (s + \frac{n-1}{2})$ , we have

$$b_P(s) = (s + \frac{n-1}{2}) \tilde{b}(s)$$

where

$$\tilde{b}(s) = s(s + \frac{1}{2}) \dots (s + \frac{n-2}{2}).$$

Then  $p = 1$  in Algorithm 9.1. We have the Taylor expansion

$$\tilde{b}(s)^{-1} = \sum_{i=0}^{\infty} b_i (s + \frac{n-1}{2})^i$$

with

$$b_0 = \left( (-\frac{n-1}{2})(-\frac{n-2}{2}) \dots (-\frac{1}{2}) \right)^{-1}.$$

Then, by the third step of the procedure in Algorithm 9.1, we have

$$u(x) = \sum_{i+j=-\lfloor \frac{n+1}{2} \rfloor + 1} b_i P_j^{[\tilde{\alpha}_0^{\#1}, -\frac{n+1}{2}+1]}(x) = \sum_{i+j=-\lfloor \frac{n-1}{2} \rfloor} b_i P_j^{[\tilde{\alpha}_0^{\#1}, -\frac{n-1}{2}]}(x)$$

Since  $i \geq 0$  and  $P_j^{[\vec{a}_0^{\#1}, -\frac{n-1}{2}]}(x) = 0$  except for  $j \geq -\lfloor \frac{n-1}{2} \rfloor$ , we have

$$u(x) = \left( \left(-\frac{n-1}{2}\right) \left(-\frac{n-2}{2}\right) \cdots \left(-\frac{1}{2}\right) \right)^{-1} \times P_{-\lfloor \frac{n-1}{2} \rfloor}^{[\vec{a}_0^{\#1}, -\frac{n-1}{2}]}(x).$$

This is a  $G$ -invariant fundamental solution to  $P(x, \partial) = \det(\partial^*)$ .

#### APPENDIX A. SOME RESULTS IN THE PREVIOUS PAPER BY THE AUTHOR [12].

The following subsections are devoted to explaining the results quoted from the author's paper [12], which play crucial roles in this paper. We shall give the statements of the theorems used in this paper for the reader's convenience without proof.

**A.1. The exact order of complex power functions.** Using the vectors  $\mathbf{d}^{(k)}[s_0]$  defined in (78), we can determine the exact orders of poles of  $P^{[\vec{a}, s]}(x)$ .

**Theorem A.1** (Exact orders of poles). *The exact orders of poles of  $P^{[\vec{a}, s]}(x)$  are computed by the following algorithm.*

1. At  $s = -\frac{2m+1}{2} (m = 1, 2, \dots)$ , the coefficient vectors  $\mathbf{d}^{(k)}[-\frac{2m+1}{2}]$  are defined in Definition 7.1. The exact order  $P^{[\vec{a}, s]}(x)$  at  $s = -\frac{2m+1}{2} (m = 1, 2, \dots)$  is given in terms of the coefficient vector  $\mathbf{d}^{(2k)}[-\frac{2m+1}{2}]$ .
  - (a) If  $1 \leq m \leq \frac{n}{2}$ , then  $P^{[\vec{a}, s]}(x)$  has a possible pole of order not larger than  $m$ .
    - If  $\langle \mathbf{d}^{(2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$ , then  $P^{[\vec{a}, s]}(x)$  is holomorphic, and the converse is true.
    - For integers  $p$  in  $1 \leq p < m$ , if  $\langle \mathbf{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$  and  $\langle \mathbf{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $p$ , and the converse is true.
    - Lastly, if  $\langle \mathbf{d}^{(2m)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $m$ , and the converse is true.
  - (b) If  $m > \frac{n}{2}$ , then  $P^{[\vec{a}, s]}(x)$  has a possible pole of order not larger than  $n' := \lfloor \frac{n}{2} \rfloor$ .
    - If  $\langle \mathbf{d}^{(2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$ , then  $P^{[\vec{a}, s]}(x)$  is holomorphic, and the converse is true.
    - For integers  $p$  in  $1 \leq p < n'$ , if  $\langle \mathbf{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$  and  $\langle \mathbf{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $p$ , and the converse is true.
    - Lastly,  $P^{[\vec{a}, s]}(x)$  has a pole of order  $n'$  if  $\langle \mathbf{d}^{(n-1)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$  (when  $n$  is odd) or  $\langle \mathbf{d}^{(n)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$  (when  $n$  is even), and the converse is true.
2. At  $s = -m (m = 1, 2, \dots)$ , the coefficient vectors  $\mathbf{d}^{(k)}[-m]$  are defined in Definition 7.1 with  $\epsilon[-m] = (-1)^{-m+1}$ . We obtain the exact order at  $s = -m (m = 1, 2, \dots)$  in terms of the coefficient vectors  $\mathbf{d}^{(2k+1)}[-m]$ .
  - (a) If  $1 \leq m \leq \frac{n}{2}$ , then  $P^{[\vec{a}, s]}(x)$  has a possible pole of order not larger than  $m$ .
    - If  $\langle \mathbf{d}^{(1)}[-m], \vec{a} \rangle = 0$ , then  $P^{[\vec{a}, s]}(x)$  is holomorphic, and the converse is true.

## HYPERFUNCTION SOLUTIONS TO INVARIANT DIFFERENTIAL EQUATIONS

- For integers  $p$  in  $1 \leq p < m$ , if  $\langle d^{(2p+1)}[-m], \vec{a} \rangle = 0$  and  $\langle d^{(2p-1)}[-m], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $p$ , and the converse is true.
  - Lastly, if  $\langle d^{(2m-1)}[-m], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $m$ , and the converse is true.
- (b) If  $m > \frac{n}{2}$ , then  $P^{[\vec{a}, s]}(x)$  has a possible pole of order not larger than  $n' := \lfloor \frac{n+1}{2} \rfloor$
- If  $\langle d^{(1)}[-m], \vec{a} \rangle = 0$ , then  $P^{[\vec{a}, s]}(x)$  is holomorphic, and the converse is true.
  - For integers  $p$  in  $1 \leq p < n'$ , if  $\langle d^{(2p+1)}[-m], \vec{a} \rangle = 0$  and  $\langle d^{(2p-1)}[-m], \vec{a} \rangle \neq 0$ , then  $P^{[\vec{a}, s]}(x)$  has a pole of order  $p$ , and the converse is true.
  - Lastly,  $P^{[\vec{a}, s]}(x)$  has a pole of order  $n'$  if  $\langle d^{(n)}[-m], \vec{a} \rangle \neq 0$  (when  $n$  is odd) or  $\langle d^{(n-1)}[-m], \vec{a} \rangle \neq 0$  (when  $n$  is even), and the converse is true.

**A.2. The exact support of complex power functions.** The exact support of  $P^{[\vec{a}, s]}(x)$  is given by the following theorem.

**Theorem A.2** (Support of the singular invariant hyperfunctions). *Let  $q$  be a positive integer. Suppose that  $P^{[\vec{a}, s]}(x)$  has a pole of order  $p$  at  $s = -\frac{q+1}{2}$ . Let*

$$P^{[\vec{a}, s]}(x) = \sum_{w=-p}^{\infty} P_w^{[\vec{a}, -\frac{q+1}{2}]}(x) (s + \frac{q+1}{2})^w \quad (134)$$

be the Laurent expansion of  $P^{[\vec{a}, s]}(x)$  at  $s = -\frac{q+1}{2}$ . The support of the Laurent expansion coefficients  $P_w^{[\vec{a}, -\frac{q+1}{2}]}(x)$  is contained in  $S$  if  $w < 0$ .

1. Let  $q$  be an even positive integer. Then the support of  $P_w^{[\vec{a}, -\frac{q+1}{2}]}(x)$  for  $w = -1, -2, \dots, -p$  is contained in the closure  $\overline{S_{-2w}}$ . More precisely, it is given by

$$\text{Supp}(P_w^{[\vec{a}, -\frac{q+1}{2}]}(x)) = \overline{\left( \bigcup_{j \in \{0 \leq j \leq n+2w \mid \langle d_j^{(-2w)}[-\frac{q+1}{2}], \vec{a} \rangle \neq 0\}} S_{-2w}^j \right)}. \quad (135)$$

2. Let  $q$  be an odd positive integer. Then the support of  $P_w^{[\vec{a}, -\frac{q+1}{2}]}(x)$  for  $w = -1, -2, \dots, -p$  is contained in the closure  $\overline{S_{-2w-1}}$ . More precisely, it is given by

$$\text{Supp}(P_w^{[\vec{a}, -\frac{q+1}{2}]}(x)) = \overline{\left( \bigcup_{j \in \{0 \leq j \leq n+2w+1 \mid \langle d_j^{(-2w-1)}[-\frac{q+1}{2}], \vec{a} \rangle \neq 0\}} S_{-2w-1}^j \right)}. \quad (136)$$

Here,  $\text{Supp}(-)$  means the support of the hyperfunction in  $(-)$ .

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